Commutative algebra WS18
Exercise set 4.

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Problem 1. [AM Ch. 3, Ex. 7] A multiplicatively closed subset S of a ring R is said to be *saturated* if

$$xy \in S \qquad \Leftrightarrow \qquad x \in S \text{ and } y \in S.$$

Prove that

(1) S saturated $\Leftrightarrow$ $R \setminus S$ is a union of prime ideals.
(2) If S is any multiplicatively closed subset of R, there is a unique smallest saturated multplicatively closed subset $\bar{S}$ containing S, and that $\bar{S}$ is the complement in R of the union of all prime ideals which do not intersect S. ($\bar{S}$ is called the *saturation* of S.)

If $S = 1 + \mathfrak{a}$ for an ideal $\mathfrak{a}$, find $\bar{S}$.

Problem 2. [AM Ch. 3, Ex. 8] Let S, T be multiplicatively closed subsets of R, such that $S \subseteq T$. Let $\phi : S^{-1}R \to T^{-1}R$ be the homomorphism which maps each $a/s \in S^{-1}R$ to a/s considered as an element of $S^{-1}T$. Show that the following statements are equivalent:

(1) ϕ is bijective.
(2) For each $t \in T$, $t/1$ is a unit in $S^{-1}R$.
(3) For each $t \in T$ there exists $x \in R$ such that $xt \in S$.
(4) $\bar{T} = \bar{S}$.
(5) Every prime ideal which intersects T also intersects S.

Problem 3. [AM Ch. 3, Ex. 6] Let R be a non-zero ring and let Σ be the set of all multiplicatively closed subsets $S \subset R$ such that $0 \notin S$. Show that Σ has maximal elements, and that $S \in \Sigma$ is maximal if and only if $R \setminus S$ is a minimal prime ideal of R.

Problem 4. [AM, Ch. 3, Ex. 9] Let D be the set of all zero-divisors of R, and let $S_0 = R \setminus D$. Show that

(1) S_0 is saturated.
(2) D is a union of prime ideals.
(3) Every minimal prime ideal of R is contained in D.

(4) S_0 is the largest multiplicatively closed subset of R for which the homomorphism $R \to S_0^{-1}R$ is injective.
(5) Every element of $S_0^{-1}R$ is either a zero-divisor or a unit.
(6) $R \to S_0^{-1}R$ is bijective if and only if every element of R is a zero-divisor or a unit.

Problem 5. [AM, Ch. 3, Ex. 5] Let R be a ring. For every prime ideal $\mathfrak{p} \subset R$, denote by $R_\mathfrak{p}$ the localization of R with respect to the multiplicative set $R \setminus \mathfrak{p}$. Suppose that, for each prime $\mathfrak{p}$, $R_\mathfrak{p}$ has no nilpotent elements $\neq 0$. Show that R has no nilpotent elements $\neq 0$. If each $R_\mathfrak{p}$ is an integral domain, is R necessarily an integral domain?

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