Commutative algebra WS18
Exercise set 1.

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Dictionary:
$x \in R$ is a *unit* if there exists $y \in R$ such that $xy = 1$.
$x \in R$ is *nilpotent* if there exists $n > 0$ integer such that $x^n = 0$.
$x \in R$ is a *zero-divisor* if there exists $y \in R$, $y \neq 0$ such that $xy = 0$.
If $x \in R$, we denote by (x), and sometimes by xR, the ideal consisting of elements of the form xy for $y \in R$.

Problem 1. [AM Ch. 1, Ex. 1] Let x be a nilpotent element of a ring R. Show that $1 + x$ is a unit of R. Deduce that the sum of a nilpotent element and a unit is a unit.

Problem 2. [AM Ch. 1, Ex. 2] Let R be a ring. Let $f = f_0 + f_1 x + \cdots f_n x^n \in R[x]$. Prove that

(1) f is a unit in $R[x]$ if and only if f_0 is a unit in R and f_i is nilpotent for $i > 0$.
(2) f is nilpotent if and only if f_i is nilpotent for all i.
(3) f is a zero-divisor if and only if there exists $a \in R$, $a \neq 0$ such that $af_i = 0$ for all i.

Problem 3. [AM Ch. 1, Ex. 3] Let R be a ring. Let $f = f_0 + f_1 x + \cdots \in R[[x]]$. Prove that

(1) f is a unit in $R[[x]]$ if and only if f_0 is a unit in R.
(2) If f is nilpotent, then f_i is nilpotent for all i. Is the converse true?

Problem 4. [AM, Prop. 1.2] Let R be a ring. Show that the following are equivalent:

(1) R is a field.
(2) $0 \neq 1$ and the only ideals of R are (0) and (1).
(3) Every homomorphism $R \to R'$ is injective for every non-zero ring R'.

Problem 5. Let k be a field and $n > 0$ an integer. Describe all ideals in the ring $k[x]/(x^n)$.

Due date: 16.10.2018