

Algebraic Topology WS26

Exercise set 2.

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Problem 1. [Hatcher, Ch. 0, Ex. 10] Show that a nonempty space X is contractible if and only if for every space Y every map $f : X \rightarrow Y$ is null-homotopic (homotopic to a constant map). Similarly, show that X is contractible if and only if for every space Y every map $f : Y \rightarrow X$ is null-homotopic.

Problem 2. [Hatcher, Ch. 0, Ex. 11] Let $f : X \rightarrow Y$. Suppose there exist $g, h : Y \rightarrow X$ such that $f \circ g \simeq \text{Id}_Y$ and $h \circ f \simeq \text{Id}_X$. Show that f is a homotopy equivalence. More generally, prove the *two-out-of-six property*:

$$Z \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} W.$$

If $f \circ g$ and $h \circ f$ are homotopy equivalences, then f , g , h and $h \circ f \circ g$ are homotopy equivalences.

Problem 3. [Hatcher, Ch. 1.1, Ex. 15] Given a map $f : X \rightarrow Y$ and a path $h : I \rightarrow X$ from x_0 to x_1 , show that $\pi_1(f)\beta_h = \beta_{f \circ h}\pi_1(f)$ in the diagram below.

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\beta_h} & \pi_1(X, x_1) \\ \downarrow \pi_1(f) & & \downarrow \pi_1(f) \\ \pi_1(Y, f(x_0)) & \xrightarrow{\beta_{f \circ h}} & \pi_1(Y, f(x_1)) \end{array}$$

Problem 4. [Hatcher, Ch. 1.1, Ex. 3] Let X be a path-connected space and $x_0 \in X$. Show that $\pi_1(X, x_0)$ is abelian if and only if all basepoint-change homomorphisms β_h depend only on the endpoints of the path h .

Problem* 5. [Hatcher, Theorem 1.10]¹ Prove the Borsuk–Ulam theorem in dimension two: for every continuous map $f : S^2 \rightarrow \mathbb{R}^2$ there is a pair of antipodal points $x, -x \in S^2$ with $f(x) = f(-x)$.

Hint. Assuming there is no such pair, consider

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|} \in S^1.$$

Restrict g to the equator and use Problem 5* of Exercise set 1.

Due date: Monday, 19.10.2026

¹Harder problems are marked with *.