Algebraic Topology SS19
Exercise set 8.

Instructor: Anton Mellit

Problem 1. [Snake lemma, Wikipedia] Suppose we have the following commutative diagram of abelian groups and maps between them:

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & \end{tikzcd}$$

Suppose the rows of the diagram are exact. Construct a homomorphism $\ker c \to \operatorname{coker} a$ so that we have an exact sequence

$$\ker a \to \ker b \to \ker c \to \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c.$$

Problem 2. [Five lemma, Wikipedia] Suppose we have a commutative diagram of abelian groups

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, "c"] & D \arrow[r, "j"] \arrow[d, "d"] & E \arrow[d, "e"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "j'"] & E' \end{tikzcd}$$

with exact rows. Suppose b and d are isomorphisms, a is surjective and e is injective. Show that c is an isomorphism.

Problem 3. Let X be a topological space with a point $*$. Construct a group homomorphism from $\pi_1(X,*)$ to H_1. Show that this homomorphism is surjective and its kernel is the commutator subgroup (the subgroup generated by commutators $[x,y]$ for $x,y \in \pi_1(X,*)$).

Problem 4. Let C be a complex with $C_1 = C_0 = \mathbb{Z}$ and $\partial_1 : C_1 \to C_0$ is given by $\partial(x) = 2x$, all other C_i are zero. Let C' be a complex with $C_0' = \mathbb{Z}/2$ and $C_i' = 0$ for $i \neq 0$. Show that the complexes C and C' have isomorphic homology, but C and C' are not homotopy equivalent.

Problem 5. Show that composition of maps of complexes respects the homotopy equivalence ($f \simeq f', g \simeq g'$ implies $f \circ g \sim f' \circ g'$).

Due date: 21.05.2019