Question:

$G = S_5$ we have the following (irreducible) reps: (dimensions)

$1, 1, 4, 4, x, x, y$

$x = ?$ $(x, y) = ?$

$y = ?$

$5! - 1^2 - 1^2 - 4^2 - 4^2 = 86 = 2x^2 + y^2$

$x = 5 \quad y = 6 \qquad 50 + 36 = 86.$

Relationship to probability theory.

Consider the problem:

Fix n, π = random permutation $\pi \in S_n$

Let $\ell(\pi)$ = number of fixed elements under π

$= \#\{i \mid \pi(i) = i\}$.

We study the random variable $\ell(\pi)$.

What is $\mathbb{E}\,\ell(\pi) = ?$

s.d. $(\ell(\pi)) = \mathbb{E}\left(\ell(\pi) - \mathbb{E}\,\ell(\pi)\right)^2$.

Solution relate to character theory:

Let V = permutation rep. of S_n ($\dim V = n$)

$V = \mathbb{C} \oplus V'$

trivial rep; orthogonal complement of $(1, 1, \ldots, 1)$

$\ell(\pi) = \chi_V(\pi)$. $\qquad \mathbb{E}\,\ell(\pi) = \frac{1}{|S_n|} \sum_{\pi \in S_n} \chi_V(\pi)$

# of times $\mathbb{C}$ appears in the decomposition of V $= \langle \chi_V, \chi_{\mathbb{C}} \rangle = \frac{1}{|S_n|} \sum_{\pi \in S_n} \chi_V(\pi) \cdot \underset{\chi_{\mathbb{C}}(\pi^{-1})}{1}$

$= 1$. So $\mathbb{E}\,\ell(\pi) = 1$.

Direct proof? Let $\delta_i(\pi) = \begin{cases} 0 & \text{if } \pi(i) \neq i \\ 1 & \text{if } \pi(i) = i \end{cases}$

then $\ell(\pi) = \sum_{i=1}^{n} \delta_i(\pi)$, so $\mathbb{E}\,\ell = \sum_{i=1}^{n} \underbrace{\mathbb{E}\,\delta_i}_{\text{prob. (} i \text{ is fixed)}} = \sum \frac{1}{n} = 1$.

s.d. ? $\quad \ell(\pi) - \underset{\mathbb{E}\ell}{1} = \chi_{V'}(\pi)$

s.d. $= \frac{1}{|S_n|} \sum_{\pi \in S_n} \chi_{V'}(\pi)^2 = \frac{1}{|S_n|} \sum \chi_{V'}(\pi) \chi_{V'}(\pi^{-1})$

$1 = \langle \chi_{V'}, \chi_{V'} \rangle$ (V' irreducible)

(Remark for $\Gamma = S_n$ $\chi_V(\pi) = \chi_V(\pi^{-1})$ (More about this later)).

This works for arbitrary group Γ if Γ acts on some set A, consider $(1, \ldots, 1)^{\perp}$ in $\mathbb{C}[A]$ = permutation rep. If it is irreducible $\Rightarrow$ s.d. $= 1$. Otherwise s.d. > 1.

Direct proof for S_n:

$\mathbb{E}\,\ell(\pi)^2 = \sum_{i,j} \mathbb{E}(\delta_i \delta_j) = \sum_{i=j} + \sum_{i \neq j}$

$\sum_{i=j} = \frac{1}{n} \cdot n = 1 \qquad \sum_{i \neq j} = \frac{1}{n(n-1)} \cdot n(n-1) = 1$

hence $\mathbb{E}\,\ell(\pi)^2 = 2$

s.d. $= \mathbb{E}\left(\ell(\pi) - 1\right)^2 = 2 - 2 \cdot 1 + 1 = 1$ correct.

Break until: 10:45.

Character tables. (Recall sage, character_table).

Fix Γ finite group.

Suppose $V_1, .., V_m$ is a complete list of irreducible reps.

Observation Denote $g \sim g'$ (g conjugate to g') if $\exists h \in \Gamma$ s.t. $g' = hgh^{-1}$. In this case $\operatorname{tr} \rho_V(g) = \operatorname{tr} \rho_V(g')$ for any rep. V.

"$\sim$" is an equivalence relation, its equivalence classes are called conjugacy classes.

Let's list all conjugacy classes $[g_1], .., [g_{m'}]$ represented by $g_1, .., g_{m'}$.

The character table is a table of $\{\chi_{V_i}(g_j)\}$.

$$\begin{pmatrix} \chi_{V_1}(g_1) & \chi_{V_1}(g_2) & \cdots \\ \chi_{V_2}(g_1) & \chi_{V_2}(g_2) & \cdots \\ \vdots & & \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1m'} \\ \vdots & & & \\ C_{m1} & & \cdots & C_{mm'} \end{pmatrix}$$

Properties of character tables:

1) first orthogonality relation (rows)
2) second —"— (columns) ?
3) $m = m'$ (# of irreducible reps = # of conjugacy classes)

We will see that $1) + 2) \Rightarrow 3)$.

First orthogonality relation:

Rows of the character table are orthogonal w.r.t to the following scalar product:

(Remember that the "correct" scalar product is

$$\langle \chi, \chi' \rangle = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi(g) \chi'(g^{-1}) = \sum_{i=1}^{m'} \frac{|[g_i]|}{|\Gamma|} \chi(g_i) \chi(g_i^{-1})$$

Introduce $\bar{i}$ by $[g_i^{-1}] = [g_{\bar{i}}]$.

$i \to \bar{i}$ is an involution:

$$\langle \chi, \chi' \rangle = \sum_{i=1}^{m'} \frac{|[g_i]|}{|\Gamma|} \chi(g_i) \chi'(g_{\bar{i}}).$$

The first orth. rel-n: $\forall i, i'$

$$\sum_{j=1}^{m'} \frac{|[g_j]|}{|\Gamma|} C_{ij} C_{i'\bar{j}} = \begin{cases} 0 & i \neq i' \\ 1 & i = i'. \end{cases}$$

$C_{ij} = \chi_{V_i}(g_j)$.

2) Second orth. rel-n. (New for us).

Observation: Let $V = \mathbb{C}[\Gamma]$.

$g, g' \in \Gamma$.

$$\sum_{i=1}^{m} \frac{1}{|\Gamma|} \chi_{V_i}(g) \chi_{V_i}(g'^{-1}) = \begin{cases} 0 & g \not\sim g' \\ 1/|[g]| & g \sim g'. \end{cases}$$

Example

$$S_3 \quad \begin{array}{c|ccc} & id & \tau & cyc \\ \hline triv & 1 & 1 & 1 \\ sign & 1 & -1 & 1 \\ V' & 2 & 0 & -1 \end{array}$$

Colums are orthogonal.

rows are orthogonal with weights $\frac{1}{6}, \frac{3}{6}, \frac{2}{6}$.

$\langle$ first col., first col. $\rangle = \frac{1}{6} \cdot 6 = 1$

$\langle$ second col., secon col. $\rangle = \frac{1}{6} \cdot 2 = \frac{1}{3}$

$\langle$ third col., third col. $\rangle = \frac{1}{6} \cdot 3 = \frac{1}{2}$.

2nd orth. relation

$$\forall j, j' \qquad \sum_{i=1}^{m} C_{ij} C_{i\bar{j'}} = \begin{cases} 0 & j \neq j' \\ \frac{|\Gamma|}{|[g_j]|} & j = j'. \end{cases}$$