Irreducible reps of $SU(2)$ are Sym^k $k=0,1,\dots$

Their characters are given by:

$g \in SU(2)$ is conjugate to $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$,

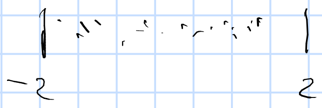
$\bar{u} = u^{-1}$

$|u| = 1$

$$\chi_{\mathrm{Sym}^k}\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = u^k + u^{k-2} + \dots + u^{-k} \quad (*)$$

Conjugacy is completely determined by $u + u^{-1} = t \in [-2, 2]$

Let $P_k(t)$ be s.t.



$P_k(u+u^{-1}) = u^k + \dots + u^{-k}$ (t)

$P_0(t) = 1$

$P_1(t) = t$

$P_2(t) = t^2 - 1$ $\qquad u^2 + 1 + u^{-2} = (u+u^{-1})^2 - 1$

$P_3(t) = t^3 - 2t$ $\qquad u^3 + u + u^{-1} + u^{-3} = (u+u^{-1})^3 - 2(u+u^{-1})$

Chebyshev polynomials:

$u + u^{-1} = 2\cos\alpha$ $\qquad u = e^{i\alpha}$

$$u^k + u^{k-2} + \dots + u^{-k} = \frac{u^{k+1} - u^{-k-1}}{u - u^{-1}} =$$

$$= \frac{\sin((k+1)\alpha)}{\sin\alpha}$$

$\sin(2\alpha) = 2\sin\alpha\cos\alpha$

$$\sin((k+1)\alpha) = P_k(\cos\alpha)\cdot\sin\alpha.$$

$P_k = U_k$ Second kind Chebyshev polynomials.

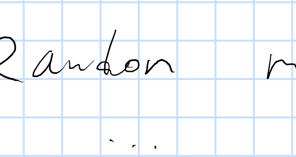
The unknown probability measure on $[-2, 2]$ is so that $P_0, P_1, \dots$ form an orthonormal basis

$$\int_{-2}^{2} P_i(t)\, P_j(t)\, \mu(dt) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$\Rightarrow$ $\mu(dt) \sim \sqrt{1-x^2}\, dx$ $\quad t = 2x$

Wigner semicircle distribution.

random point on disc.

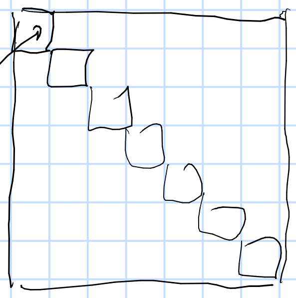


density

$\Rightarrow$ Random matrix theory, ...

Exercise

take a random unitary matrix of size n



x_{11}

$x_{11}(g)$

Question: average $|x_{11}(g)|^2 = ?$

Solution:

$$\text{average}\left|\sum_i x_{ii}(g)\right|^2 = 1 \qquad \left(\sum x_{ii} \text{ is the character}\right)$$

$$= E\left(\sum_i x_{ii}(g) \sum_i \overline{x_{ii}(g)}\right) =$$

$$= E\left(\sum_i |x_{ii}(g)|^2 + \underbrace{\sum_{i \neq j} x_{ii}(g)\overline{x_{jj}(g)}}_{=0}\right)$$

Because we can multiply the first row by -1, the third row by -1

$\Downarrow$

$E\, x_{11}(g)\overline{x_{22}(g)} = 0$

$\Downarrow$

$$E|x_{11}(g)|^2 = \frac{1}{n}.$$

Ex Consider the group $SO(3)$ orientation-preserving rotations of $\mathbb{R}^3$.

Basic facts:

$SU(2)$ acting on homogeneous polynomials of degree 2 (3 dimensional space) by certain choice of basis, we can make all matrices real $\Rightarrow$ homomorphism $SU(2) \to SO(3)$. Surjective.

kernel $= \pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

irreducible representations of $SO(3)$

$SU(2) \to SO(3) \to U(n)$

induce irreducible reps of $SU(2)$ such that $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ acts as identity, so we obtain reps $Sym^0, Sym^2, Sym^4, \ldots$ (only even ones).

$\Rightarrow$ we can compute integrals over $SO(3)$.

General Theorem (Peter-Weyl)

If G is a compact group.

Consider $C^\infty(G)^G$ = continuous functions f on G such that $f(gxg^{-1}) = f(x)$ (invariant under conjugation)

Define scalar product by
$$(f_1, f_2) = \int_G f_1(g)\overline{f_2(g)}\,\mu(dg)$$
Haar measure

norm by $\sqrt{(f,f)}$.

Let $L_2(G)^G$ be the completion of $C^\infty(G)^G$ w.r.t. this norm.

Theorem: Characters of irreducible reps form an orthonormal basis

We have proved that they are orthonormal, so the theorem says there are no functions in
$$(\chi_1, \chi_2, \ldots, \chi_5, \ldots)^\perp$$
all characters

orthogonal complement.

$\Rightarrow$ any function $f \in C^\infty(G)^G$ can be written as Fourier series
$$f = \sum_{i=1}^\infty (f, \chi_i)\,\chi_i .$$

Question: what about all functions?

Answer: Similarly define
$C(G)$ = continuous functions on G
$L^2(G)$ = completion.

Theorem (continuation)

Take all irreducible representations.

reps	chars				
ρ_1	χ_1	$\rho_1(g)_{11}$	$\rho_1(g)_{12}$	$\ldots$	$\rho_1(g)_{d_1 d_1}$
ρ_2	χ_2	$\rho_2(g)_{11}$	$\rho_2(g)_{12}$	$\ldots$	$\rho_2(g)_{d_2 d_2}$
ρ_3	χ_3				

make every rep unitary.

Consider $\rho_i(g)$ = matrix $d_i \times d_i$, some d_i

$\rho_i(g)_{kl}$ = number in the box (k,l).

$\{\rho_i(g)_{kl}\} \in C(G)$ are an orthogonal basis of $L_2(G)$.

For $SU(2)$

Sym^0 $\quad 1$

Sym^1 $\quad g_{11}\ g_{12}\ g_{21}\ g_{22}$

$\vdots$

Sym^k $\quad (k+1)^2$ functions

$$|\rho_i(g)_{kl}|^2 = \frac{1}{d_i}$$

Break ~ 10:36

Exam Make sure you are registered for the course

I'll send my phone by email.

Registration: send me an email before Friday (26 - June)

"hi, I want to register".

Exam 26 June
9:45 – 11:15
On mellit.org/teaching/rep.theory
pdf with questions.
+ My zoom will be on, no need to be in zoom.

Solutions: before 11:15 (+5 minutes gap for sending)
by email (Photo) or scan e.g. camscanner app

Rules: use nothing
(no google, books, wikipedia).
no contacting others)

~ 10 questions of "short" type.

All reps before:
indecomposable $\Rightarrow$ irreducible.

(V, ρ_V) $\to$ indecomposable $\to$ $\to$ finished.

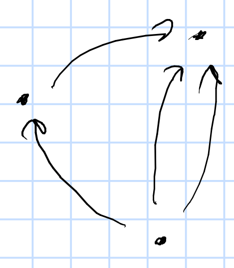
This effect is called semisimplicity.

There is a notion of semisimple category, the categories of representations of finite, compact groups are semisimple.
"Classical rep theory".

Modern rep. theory deals with non-semisimple case.

<u>Basic</u> <u>example</u> Category of reps of a quiver.

Fix a <u>quiver</u> (= oriented graph) $Q = (V, E)$ vertices, arrows (edges)



<u>Def</u> A representation of a quiver is a collection of vector spaces $\{X_v\}_{v \in V}$ and linear maps $\{m_e\}_{e \in E}$ where if $e: v \to v'$

$\underset{v}{\bullet} \xrightarrow{e} \underset{v'}{\bullet}$

then $m_e: X_v \to X_{v'}$.

Consider these reps up to change of basis of every X_v.

<u>Example</u> 1) $\bullet \circlearrowleft$ = X_v, $m: X_v \to X_v$ up to conjugation.

2) $\bullet \longrightarrow \bullet$

Study triples X_1, X_2, $m: X_1 \to X_2$.

Already in example 1 we have Jordan form theorem.

A subrepresentation is a collection of subspaces $Y_v \subset X_v$ all $v \in V$ so that $m_e(Y_v) \subset Y_{v'}$ for $e: v \to v'$.

A representation is a direct sum if $X_v = Y_v \oplus \tilde{Y}_v$ for subrepresentations $\{Y_v\}$, $\{\tilde{Y}_v\}$

Consider representation

$\underset{\mathbb{C}}{\bullet} \xrightarrow{id} \underset{\mathbb{C}}{\bullet}$ is indecomposable. (no non-trivial direct sum decomposition)

is reducible

$\underset{0}{\bullet} \xrightarrow{0} \underset{\mathbb{C}}{\bullet}$ $\swarrow$ is a non-trivial sub
$\cap \qquad \cap$
$\underset{\mathbb{C}}{\bullet} \longrightarrow \underset{\mathbb{C}}{\bullet}$

try to write

$$\begin{matrix} \mathbb{C} \xrightarrow{0} 0 \\ \oplus \\ 0 \xrightarrow{0} \mathbb{C} \end{matrix}$$

we don't get $\mathbb{C} \xrightarrow{id} \mathbb{C}$.

First example of non-semisimple case.

$0 \to X_2$

$X_1 \to X_2$

trivial sub: either all $Y_v = 0$ or all $Y_v = X_v$.

Still study indecomposable reps.

For instance

$\underset{X_1}{\bullet} \xrightarrow{m} \underset{X_2}{\bullet}$

$X_1 = \operatorname{Ker} m \oplus Y$
$X_2 = \underset{\operatorname{Im} m}{\underset{\parallel}{Y}} \oplus Z$

So we have a direct sum decomposition into 3 reps.

$\operatorname{Ker} m \to 0$
$Y \xrightarrow{\sim} Y$
$0 \to Z$

So we have 3 indecomposable reps:

$\mathbb{C} \to 0$
$0 \to \mathbb{C}$
$\mathbb{C} \xrightarrow{id} \mathbb{C}$

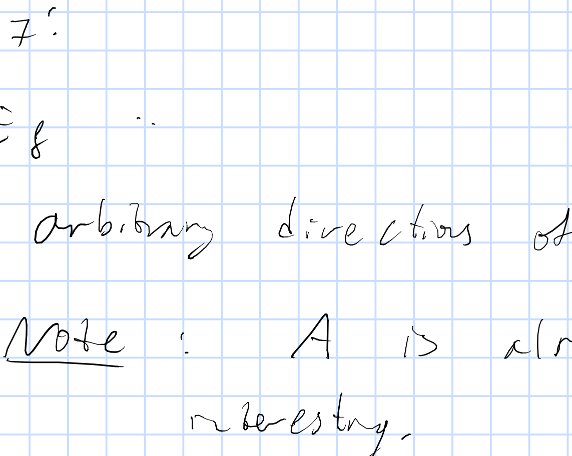
Complete answer here.

For which quivers do we get a finite list?
Answer: Dynkin diagrams.

A:

D:

E_6:



E_7:

E_8

arbitrary directions of arrows.

<u>Note</u>: A is already interesting.

Answer: indecomposables are

$0 \quad 0 \dots 0 \quad \mathbb{C} \xrightarrow{id} \mathbb{C} \xleftarrow{id} \dots \xrightarrow{id} \mathbb{C} \quad 0 \dots 0$

(Exercise)