Today: unitary representations.

Remark So far everything was algebraic, we used only operations $+, -, \cdot$, division by $|\Gamma|$.
Jordan decomposition (field is algebraically closed) $\Rightarrow$ many statements are still true, for instance over $\overline{\mathbb{Z}/p\mathbb{Z}}$ & algebraic closure, p prime, if $p \nmid |\Gamma|$.

note: in this field we don't have the notion of >0.

Over $\mathbb{C}$ we have complex conjugation, we have $\mathbb{R} \subset \mathbb{C}$, on $\mathbb{R}$ we have $>$ (order).

Recall: Let V be a $\mathbb{C}$-vector space. A hermitian form on V is an operation $(\cdot,\cdot): V \times V \to \mathbb{C}$, satisfying

$(\alpha x, y) = \alpha (x,y)$ $\quad x, y, z \in V$
$(x+y, z) = (x,z) + (y,z)$ $\quad \alpha \in \mathbb{C}$
$\overline{(x,y)} = (y,x)$.
complex conjugation

In particular (x,x) is in $\mathbb{R}$ $\forall x \in V$.
The form is called *positive definite* if $(x,x) \geq 0$ $\forall x \in V$ and $(x,x) = 0 \Leftrightarrow x = 0$.
We say $\sqrt{(x,x)}$ is the length of x.

Example: $V = \mathbb{C}^n$,
$$(x,y) = \sum_{i=1}^{n} x_i \overline{y_i} = \overline{y^t} x$$
$$(\overline{y_1}, \overline{y_2}, \dots, \overline{y_n}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x,y)$$
Arbitrary hermitean form can be written as $\overline{y^t} A x$ for a matrix A s.t. $\overline{A^t} = A$ (notation: $A^* = \overline{A^t}$) $A^* = A$.

note: viewing $\mathbb{C}^n$ as $\mathbb{R}^{2n}$, we get the same distance, since $x_i = a_i + b_i\sqrt{-1}$,
$$(x,x) = \sum_{i=1}^{n} x_i \overline{x_i} = \sum_{i=1}^{n} a_i^2 + b_i^2.$$

Remark if V is a real vector space with euclidean form, then there is a corresponding complex vector space $(V \otimes_{\mathbb{R}} \mathbb{C})$ whose elements are formal combinations $v^{(1)} + \sqrt{-1}\, v^{(2)}$ for $v^{(1)}, v^{(2)} \in V$.
euclidean form on V induces a hermitean form on $V \otimes_{\mathbb{R}} \mathbb{C}$ defined in a straightforward way.
(A is a matrix of a euclidean form $\Rightarrow$ $A^t = A$, $\overline{A} = A \Rightarrow A^* = A$)

Suppose V has a hermitean form
Def $U: V \to V$ is unitary if $(Ux, Ux) = (x,x)$.

a representation (V, ρ_V) is unitary if $\forall g \in \Gamma$ $\rho_V(g)$ is unitary.

Def (V, ρ_V) is a representation, is called unitarizable if $\exists$ a positive definite hermitean form on V for which it is unitary.

Theorem 1 Every finite dimensional rep of a finite group is unitarizable
Theorem 2 In the case of irreducible representation in a unique way (up to a scalar multiple)

Pf

1) Let $(\cdot,\cdot)'$ be some positive definite hermitean form.
Let
$$(x,y) = \sum_{g \in \Gamma} (\rho_V(g)x, \rho_V(g)y)'.$$
Clearly $(x,x) = \sum$ positive terms > 0, $x \neq 0$.
(V, ρ_V) is now unitary.

2) Suppose we have 2 different forms. $(\cdot,\cdot)_1$, $(\cdot,\cdot)_2$
$\exists$ a matrix $A: V \to V$ s.t.
$$(x,y)_2 = (Ax, y)_1.$$
Claim: $A\rho_V(g) = \rho_V(g)A$ $\forall g \in \Gamma$.
Pf:
$$(A\rho_V(g)x, y)_1 = (\rho_V(g)x, y)_2 = (x, \rho_V(g)^{-1}y)_2 = (Ax, \rho_V(g)^{-1}y)_1 = (\rho_V(g)Ax, y)_1$$
holds for all $y \Rightarrow A\rho_V(g)x = \rho_V(g)Ax$
holds for all $x \Rightarrow A\rho_V(g) = \rho_V(g)A$.
By Schur's lemma $\exists \lambda \in \mathbb{C}$ s.t. $A = \lambda I$. Hence
$(x,y)_2 = \lambda (x,y)_1 \Rightarrow \lambda$ a positive real number.

We had $\chi(g^{-1})$ in orthogonality relations, for instance
$$\frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_{V_1}(g)\chi_{V_2}(g^{-1}) = \begin{cases} 0 & V_1 \neq V_2 \\ 1 & V_1 \cong V_2 \end{cases}$$

Observation for any representation (V, ρ_V) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

Pf 1 $\rho_V(g)$ can be diagonalized, so that on the diagonal we have roots of unity $e^{2\pi i \frac{k}{|\Gamma|}}$ $k \in \mathbb{Z}$
$\Rightarrow \rho_V(g^{-1})$ will have eigenvalue $e^{2\pi i\left(-\frac{k}{|\Gamma|}\right)}$ where $\rho_V(g)$ had $e^{2\pi i \frac{k}{|\Gamma|}}$,
also $e^{2\pi i\left(-\frac{k}{|\Gamma|}\right)} = \overline{e^{2\pi i k/|\Gamma|}}$ $\Rightarrow$ the sum, $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

Conclusion:
$$\frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_{V_1}(g)\overline{\chi_{V_2}(g)} = \begin{cases} 0 \\ 1 \end{cases}$$
So characters are orthogonal with respect to the standard hermitean form.

Def Element $g \in \Gamma$ is called real if g is conjugate to g^{-1}.
Prop g is real if and only if for all irreducible characters χ $\chi(g) \in \mathbb{R}$.
Pf g, g^{-1} are in the same conjugacy class $\Leftrightarrow$
$\chi(g) = \chi(g^{-1})$ for all χ.
$\Leftrightarrow$ $\chi(g) = \overline{\chi(g)}$.

Example in S_n, quaternions, D_4, all elements were real. So character tables are real.
in $\mathbb{Z}/m\mathbb{Z}$ $(m \geq 3)$, $Heis_p$ $(p \neq 2)$ not all elements are real.

Another proof of $\chi(g^{-1}) = \overline{\chi(g)}$
unitary matrix U satisfies
$$U^*U = I \Rightarrow U^{-1} = U^*$$
$$tr(U^{-1}) = tr(U^*) = tr(\overline{U^t}) = \overline{tr(U)} = tr(\overline{U})$$

10:41

Compact groups

Def A compact group Γ is a topological space and a group such that multiplication $\Gamma \times \Gamma \to \Gamma$, inverse $\Gamma \to \Gamma$ are continuous and Γ is compact.

Examples Recall $X \subset \mathbb{R}^n$ is compact if $X = \overline{X}$ (closed) and bounded.

By imposing equations on matrix groups we can construct many examples.

x) $GL_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$ not closed $\Rightarrow$ not compact.

$SO_n(\mathbb{R})$ is compact

x) $SO_n(\mathbb{R}) = \left\{ g \in GL_n(\mathbb{R}) \text{ s.t. } \begin{matrix} g^t g = I \\ \det g = 1 \end{matrix} \right\}$

Since $SO_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$ is given by continuous equations, it is closed. $\underbrace{g = (v_1, \dots, v_n)}_{\in SO_n(\mathbb{R})}$ $v_i \in \mathbb{R}^n$

$|v_i| = 1 \Rightarrow |g|^2 = n \Rightarrow SO_n(\mathbb{R})$ is bounded.

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1) $O_n(\mathbb{R}) = \{ g \in GL_n(\mathbb{R}) \quad g^t g = I \}$

2) $SL_n(\mathbb{R}) = \{ g \mid \det g = 1 \}$

3) $Sp_{2n}(\mathbb{R})$: $\mathcal{J} := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

$$Sp_{2n}(\mathbb{R}) = \{ g \mid g \mathcal{J} g^t = \mathcal{J} \}$$

4) $GL_n(\mathbb{C}) = \{ g \in \mathbb{C}^{n \times n} \mid \det g \neq 0 \}$

5) $U_n(\mathbb{C}) = \{ g \in GL_n(\mathbb{C}) / g^* g = I_n \}$

6) $SU_n(\mathbb{C}) = \{ g \in U_n(\mathbb{C}) / \det g = 1 \}$

7) $SO(m, n, \mathbb{R})$ $m > 0$ $n > 0$

$$A = \begin{pmatrix} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{m} & 0 \\ 0 & \overbrace{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}}^{n} \end{pmatrix} \quad \left\{ g \in GL_n(\mathbb{R}) / \begin{matrix} g A g^t = A \\ \det g = 1 \end{matrix} \right\}$$

Which are compact?

Answers in chat. 11:02

$SL_n \ni \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $x \to \infty$

$Sp_{2n}(\mathbb{R})$ symplectic group.

$$g = \begin{pmatrix} I & \lambda \cdot I \\ 0 & I \end{pmatrix} \in Sp_{2n} \qquad g \mathcal{J} g^t = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$n = 1$: $Sp_2(\mathbb{R}) = SL_2(\mathbb{R})$

$SO(1,1)$ columns satisfy

$\begin{pmatrix} x \\ y \end{pmatrix}$ $\quad x^2 - y^2 = 1 \leftarrow$ unbounded

instead of

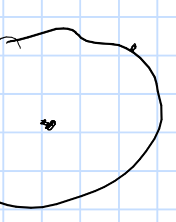
$x^2 + y^2 = 1 \Rightarrow$ bounded

$$SO(2) = \left\{ \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \right\}$$

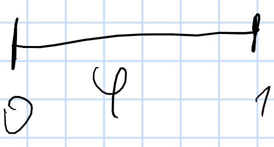
$$SO(1,1) = \left\{ \begin{pmatrix} \operatorname{ch}\varphi & \operatorname{sh}\varphi \\ -\operatorname{sh}\varphi & \operatorname{ch}\varphi \end{pmatrix} \right\}.$$

From representation theory point of view many tricks that worked for finite groups also work for compact groups (for continuous representations) replacing $\sum_{g \in \Gamma}$ by $\int_{g \in \Gamma} \dots \mu(dg)$

$\uparrow$ so-called Haar measure.

Ex $U(1) \approx$ 

$e^{2\pi i \varphi}$ $\uparrow \downarrow$



0 φ 1

$\uparrow$

usual measure

$\downarrow$

Fourier analysis.