Quaternions

$\pm 1, \pm i, \pm j, \pm k$

$i^2 = -1$

As abstract group, $1 \to e$
$-1 = c$ as a set, the group consists of
$e, c, i, j, k, ci, cj, ck$
with relations:
c is central, $c^2 = e$
$i^2 = j^2 = k^2 = c$, $ij = k$,
and that's all.

$ji = ?$ $\underset{\overset{\shortparallel}{k}}{ij} \cdot ji = ici = ci^2 = \underset{\overset{\shortparallel}{e}}{c^2}$

$k \cdot ji = e$ $ji = k^{-1}$

$k^2 = c$ $k \cdot (kc) = e$

$k^{-1} = kc$

$ji = kc = ck = cij$

$jk = \underbrace{ji}j = cij^2 = i$

$ki = j$

$$\begin{tikzcd} & j \arrow[d] \\ i \arrow[ur, bend left] & k \arrow[l] \end{tikzcd}$$

multiplying along arrow gives the third
multiplying backwards gives c. third.

irreducible reps – ?
dimensions = ?
$2^2 + 1 + 1 + 1 + 1 = 8$.
The 2-dim rep:

Pauli matrices (multiplication by quaternions in the basis $1, j$) $(a+bi) + j(c+di)$

$$\begin{pmatrix} & \\ & \end{pmatrix}$$

$e \longrightarrow \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$

$i \longrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$j \longrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$k \to \begin{pmatrix} & -i \\ -i & \end{pmatrix}$

$c \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

i^2, j^2, k^2 go to -1
ij go to k.

why not
$(a+bi) + (c+di)j$?

$Q =$ Quaternions
$\shortparallel$
$\mathbb{C}^2$

$$\begin{tikzcd} Q \arrow[r, "\text{acts}"'] \arrow[d, Rightarrow] & Q \arrow[d, equal] \\ j \arrow[r, hook, "\text{matrix}"'] & \mathbb{C}^2 \end{tikzcd}$$

every matrix commutes with multiplication by i, but j doesn't commute with i!
However multiplication by something on the left commutes with multiplication by something on the right!
$((ax)b) = (a(xb))$ (associativity).
So if we interpret the vector multiplication by i as the right multiplication we are fine, so we use presentation $(a+bi) + j(c+di)$
right multiplication

Character table

	e	c	i	j	k	ci	cj	ck
2-dim	2	-2	0	0	0	0	0	0

What about 1-dim reps?
= solutions to the equations
$i^2 = c$ $j^2 = c$ $k^2 = c$ $ij = k$ in
Complex numbers $c^2 = 1$

$c \to \pm 1$ (in fact, since $ji = cij \Rightarrow c = 1$)

$i^2 = 1$ $j^2 = 1$ $k^2 = 1$
$ij = k$

4 representations

i	j	k
1	1	1
1	-1	-1
-1	1	-1
-1	-1	1

complete the char. table:

	e	c	i	j	k	ci	cj	ck
2-dim	2	-2	0	0	0	0	0	0
1-dim reps	1	1	1	1	1	1	1	1
	1	1	1	-1	-1	1	-1	-1
	1	1	-1	1	1	-1	1	1
	1	1	-1	-1	1	-1	-1	1

note: i is conjugate to ci
j —||— cj
k —||— ck

Conjugacy classes can be represented by e, c, i, j, k:

	e	c	i	j	k
2-dim	2	-2	0	0	0
1-dim reps	1	1	1	1	1
	1	1	1	-1	-1
	1	1	-1	1	-1
	1	1	-1	-1	1

Columns are orthogonal
rows are "weighted" orthogonal,
for instance
$(1 \quad 1 \quad 1 \; 1 \; 1)$
$(1 \quad 1 \quad 1 \; -1 \; -1)$

$1 + 1 + 2 - 2 - 2 = 0$.

sizes of the conjugacy classes.

Compare with D_4:

	e	$\circlearrowright^{180^\circ}$	$\circlearrowright^{90^\circ}$	α	β
2-dim	2	-2	0	0	0
1-dim	1	1	1	1	1
	1	1	-1	1	-1
	1	1	-1	-1	1
	1	1	1	-1	-1

$\alpha\beta = \circlearrowright^{90^\circ}$

Heisenberg group after the break.

– 10:39.

The Heisenberg group over

$\mathbb{Z}/p\mathbb{Z}$.

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad a, b, c \in \mathbb{Z}/p\mathbb{Z}.$$

group of order p^3,

so it can have p-dimensional reps and 1-dimensional.

How to count 1-dimensional reps?

New idea: abelianization.

For each group G there exists an abelian G_{ab}, together with a surjective homomorphism $G \to G_{ab}$ and such that G_{ab} is as large as possible.

Def $G_{ab} = G/$subgroup generated by $aba^{-1}b^{-1}$ $\forall a, b \in G$.

Remark this is a normal subgroup because $\forall h \in G$

$$h\, ab a^{-1} b^{-1} h^{-1} = (hah^{-1})(hbh^{-1})(hah^{-1})^{-1}(hbh^{-1})^{-1}$$

in G_{ab} $\quad aba^{-1}b^{-1} = e$, so

$ab = ba \quad \forall a, b \in G_{ab}$.

Prop

1-dim reps of G are in bijection with 1-dim reps of G_{ab}.

Pf 1-dim rep G is a homomorphism

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & \mathbb{C}^* = (\mathbb{C} \setminus \{0\}) \\ G_{ab} \arrow[ur, "f'"'] & \end{tikzcd}$$

Since $\mathbb{C}^*$ is abelian, $aba^{-1}b^{-1}$ go to 1 $\forall a, b \in G$. Hence f factors through G_{ab} ($\exists f' : G_{ab} \to \mathbb{C}^*$ which makes the diagram commutative)

$\{1\text{-dim } G \text{ reps}\} \to \{1\text{-dim } G_{ab}\text{-reps}\}$

$\leftarrow$ is clear ▨

# of 1-dim reps of G_{ab}

||

# of irreducible reps of G_{ab}

||

# of conjugacy classes of G_{ab}

||

# of elements of G_{ab}.

How to compute ?

$(H_p)_{ab}$

Heisenberg group

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

Let $E_{12} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$E_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_{13} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

More explicitly, if we remember that any abelian group is a product of cyclic groups Γ abelian

$\Gamma = (\mathbb{Z}/m_1\mathbb{Z}) \times \dots \times (\mathbb{Z}/m_k\mathbb{Z})$

elements of Γ are tuples

$a_1, a_2, \dots, a_k$

$0 \leq a_1 < m_1$

$\vdots$

$0 \leq a_i < m_i$

$(a_1, a_2, \dots, a_k) \cdot (b_1, \dots, b_k) =$
$(a_1 + b_1 \bmod m_1, \dots, a_k + b_k \bmod m_k)$.

Let $x_i := (0, \dots, \underset{\substack{\uparrow \\ \text{position } i}}{1}, \dots 0)$

In a 1-dim rep x_i will go to m_i-th root of unity.

$x_i \to e^{\frac{2\pi i}{m_i} c_i}$. So a 1-dim rep of Γ is specified by the numbers $c_1, c_2, \dots, c_k$

$0 \leq c_i < m_i$. Their number is $m_1 \cdot m_2 \cdots m_k = |\Gamma|$.

$$E_{12}^a = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)^a = \left(\begin{array}{c|c} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^a & 0 \\ \hline 0 & 1 \end{array}\right) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly

$$E_{23}^c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

$E_{12}^a E_{23}^c =$ not optimal

$$E_{23}^c E_{12}^a = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

$$(E_{13})^b = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

finally

$$E_{23}^c E_{12}^a E_{13}^b = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \text{ so}$$

E_{12}, E_{13}, E_{23} generate H_p.

$$\underbrace{E_{12}^a E_{23}^c}_{} = \begin{pmatrix} 1 & a & ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{E_{23}^c E_{12}^a}_{} E_{13}^{ac}$$

So in fact H_p is generated by E_{12}, E_{23}.

So $(H_p)_{ab}$ is generated by the images of E_{12}, E_{23},

already in H_p $E_{12}^p = 1$ $E_{23}^p = 1$

in $(H_p)_{ab}$ we have $E_{12} E_{23} = E_{23} E_{12}$,

So maybe $(H_p)_{ab}$ is $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

the map $H_p \to (H_p)_{ab}$ is given by

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \to (a, c) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}.$$

Is this a homomorphism?

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+a' & * \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{pmatrix}$$

Indeed, it is!

$$\begin{tikzcd} H_p \arrow[r] \arrow[d] & \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \\ (H_p)_{ab} \arrow[ur, "\text{must be an isomorphism}"'] & \end{tikzcd}$$

p^2 1-dimensional reps of H_p

So there must be $p-1$ p-dim reps.

$$p^2 + (p-1)p^2 = p^3 = |H_p|.$$