F some object, assume it is always possible to embed F into an injective object, $F \subset I_0$, embed I_0/F into I_1, and so on

$0 \to F \to I_0 \to I_1 \to I_2$ exact sequence

$I_0 \to I_1 \to I_2$ is called injective resolution of F

$R^i\pi_* F := H^i(\pi_* I_0 \to \pi_* I_1 \to \dots)$.

Suppose we have a map $F \to G$.

Prop 1 Claim: this map extends to the resolutions

$$\begin{array}{ccccccccc} 0 \to & F & \to & I_0 & \to & I_1 & \to & I_2 & \dots \\ & \downarrow f & & & & & & & \\ 0 \to & G & \to & J_0 & \to & J_1 & \to & J_2 & \dots \end{array} \qquad \begin{array}{ccccc} 0 \to & F & \to & I_0 \\ & \downarrow f & & \\ 0 \to & G & \to & J_0 \end{array}$$

since J_0 is injective, we can extend $F \to G \to J_0$ to a map $I_0 \to J_0$,

so we obtain a commutative diagram

$$(*) \quad \begin{array}{ccccccc} 0 \to & F & \to & I_0 & \to & I_0/F \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & G & \to & J_0 & \to & J_0/G \end{array}$$

So we have a map $I_0/F \to J_0/G$. again these are subobjects of I_1, J_1 continue this to obtain maps $I_k \to J_k$ for all k, which make

$$\begin{array}{ccc} I_k & \to & I_{k+1} \\ \downarrow & & \downarrow \\ J_k & \to & J_{k+1} \end{array} \quad \text{commutative } \forall k.$$

In particular, applying π_* we obtain a commutative diagram

$$\begin{array}{ccccc} \pi_*(I_0) & \to & \pi_*(I_1) & \to & \dots \\ \downarrow & & \downarrow & & \\ \pi_*(J_0) & \to & \pi_*(J_1) & \to & \dots \end{array}$$

this induces maps on cohomology, so we obtain maps $R^k\pi_* F \to R^k\pi_* G \quad \forall k$.

Prop 2 Suppose we extended f in two different ways, taking difference we obtain a map of resolutions which extends 0 map $F \to G$:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "0"] & I_0 \arrow[r] \arrow[d] & I_1 \arrow[r] \arrow[d] & I_2 \arrow[d] \\ G \arrow[r] & J_0 \arrow[r] & J_1 \arrow[r] & J_2 \end{tikzcd}$$

$$\begin{tikzcd} F \arrow[r] \arrow[d, "0"'] & I_0 \arrow[r] \arrow[d, "f_0"] & I_0/F \arrow[d, dashed] \arrow[dl, dashed, "h_0"'] \\ G \arrow[r] & J_0 \arrow[r] & J_0/G \end{tikzcd}$$

$$\begin{tikzcd} I_0 \arrow[r] \arrow[d, "f_0"'] & I_0/F \arrow[r] \arrow[dl, "h_0"] & I_1 \arrow[dll, bend left, "\tilde{h}_0"] \\ J_0 & & \end{tikzcd}$$

$$\begin{tikzcd} I_0/F \arrow[r] \arrow[dr, "h_0"'] & I_1 \\ & J_0 \end{tikzcd}$$

J_0 is injective $\Rightarrow$ h_0 can be extended to $\tilde{h}_0 : I_1 \to J_0$

$$f_0 = I_0 \to I_1 \xrightarrow{\tilde{h}_0} J_0$$

now consider the next step

$$\begin{tikzcd} I_0 \arrow[r] \arrow[d, "f_0"'] & I_1 \arrow[d, "f_1"] \arrow[dl, "h_0"] \\ J_0 \arrow[r] & J_1 \end{tikzcd}$$

we will show that $\exists$ maps $h_k : I_{k+1} \to J_k$ such that $f_k = h_k d_k^I + d_{k-1}^J h_{k-1}$

$$\begin{tikzcd} & I_k \arrow[r, "d_k^I"] \arrow[dl, "h_{k-1}"'] \arrow[d, "f_k"] & I_{k+1} \arrow[dl, "h_k"] \\ J_{k-1} \arrow[r, "d_{k-1}^J"] & J_k & \end{tikzcd}$$

By induction.

$k=0$ is clear.

Now suppose we have h_k

Let's construct h_{k+1}

we know: $d_k^J f_k = f_{k+1} d_k^I$

$f_k = h_k d_k^I + d_{k-1}^J h_{k-1}$

$$\Rightarrow f_{k+1} d_k^I = d_k^J (h_k d_k^I + d_{k-1}^J h_{k-1}) = d_k^J h_k d_k^I$$

$d_k d_{k-1} = 0$

So $\quad (\underbrace{f_{k+1} - d_k^J h_k}_{g_{k+1}})\, d_k^I = 0$

we need to show $\quad g_{k+1} = h_{k+1} d_{k+1}^I$.

$$\begin{tikzcd}
I_k \arrow[r, two heads] \arrow[d, "0"] & I_k/K \arrow[r, hook, "d"] \arrow[d, "0"] & I_{k+1} \arrow[r, two heads] \arrow[d, "g_{k+1}"] & I_{k+1}/(I_k/K) \arrow[r, hook] \arrow[dl, dashed, "h_{k+1}"] & I_{k+2} \\
J_k \arrow[r, two heads] & J_k/K' \arrow[r, hook] & J_{k+1} & &
\end{tikzcd}$$

we see that h_{k+1} exists by injectivity of J_{k+1}.

The sequence $h_0, h_1, \ldots$ is called homotopy contracting $\{f_k\}$

$$f_k = h_k d_k^I + d_{k-1}^J h_{k-1} .$$

Prop 2: contracting homotopy exists if $f_0, f_1, \ldots$ extends the zero map.

Now we apply π_*

$$\begin{tikzcd}
\pi_*(I_{k-1}) \arrow[r] \arrow[d, "\pi_*(f_{k-1})"] & \underset{(1)}{\pi_*(I_k)} \arrow[r, "d"] \arrow[d, "\pi_*(f_k)"] \arrow[dl, "\pi_*(h_{k-1})"] & \pi_*(I_{k+1}) \arrow[d, "\pi_*(f_{k+1})"] \arrow[dl, "\pi_*(h_k)"] \\
\pi_*(J_{k-1}) \arrow[r] & \pi_*(J_k) \arrow[r] & \pi_*(J_{k+1})
\end{tikzcd}$$

Claim: $\pi_*(f_k)$ induces 0 map on the cohomology.

Start with x in (1) s.t. $d(x) = 0$

$$\pi_*(f_k)(x) = \big(d\, \pi_*(h_{k-1}) + \pi_*(h_k)\, \underline{d}\big) x = d\, \pi_*(h_{k-1})(x)$$

So $(2) = \pi_*(h_{k-1})(x)$ is in the image of d, so is zero in $H_k(\pi_*(J_\bullet))$.

So we proved that map $f : F \to G$ induces a map $R^k\pi_* f : R^k\pi_* F \to R^k\pi_* G$ which doesn't depend on the choices.

$F \xrightarrow{f} G \xrightarrow{g} H$ we can exten f to get (f_k) extend g to get (g_k), we can use $g_k \circ f_k$ as extension of $g \circ f$, so we see

$$R^k\pi_*(gf) \quad R^k\pi_* F \to R^k\pi_* H = (R^k\pi_* g)(R^k\pi_* f)$$

So $R^k\pi_*$ is a functor.

In particular, if F is isomorphic to F' then $R^k\pi_* F$ is isomorphic to $R^k\pi_* F'$.

Main property of derived functors is long exact sequence.

Recall that motivation for $R^1\pi_*$ was that wanted to understand for SES

$0 \to F \to G \to H \to 0$

the failure of the resulting sequence

$0 \to \pi_*(F) \to \pi_*(G) \to \pi_*(H) \to 0$ to be exact, in other words

$$0 \to \pi_*(F) \to \pi_*(G) \to \pi_*(H) \to K \to 0$$

what is K? for injective G K corresponds to $R^1\pi_* F$.

in general

<u>Prop</u> we have a long exact sequence

$\pi_* F = R^0\pi_* F$

$$0 \to \pi_* F \to \pi_* G \to \pi_* H \to R^1\pi_* F \to R^1\pi_* G \to R^1\pi_* H \to R^2\pi_* F \to \cdots$$

<u>Claim</u> We have

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$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d, hook] & G \arrow[r] \arrow[d, hook, "\exists"] \arrow[dl, dashed] & H \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & I_0 \arrow[r] \arrow[d] & I_0 \oplus J_0 \arrow[r] \arrow[d] & J_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_0/F \arrow[r] & (I_0 \oplus J_0)/G \arrow[r] & J_0/H \arrow[r] & 0
\end{tikzcd}$$

) exact, continue

snake lemma

6-lemma?

for a SES $0 \to F \to G \to H \to 0$
we have "compatible resolutions"

$$
\begin{aligned}
&F \to I_0 \to I_1 \to \cdots \\
&G \to I_0 \oplus J_0 \to I_1 \oplus J_1 \to \cdots \\
&H \to J_0 \to J_1 \to \cdots
\end{aligned}
$$

applying π_* we still have

$$\pi_*(I_k \oplus J_k) = \pi_*(I_k) \oplus \pi_*(J_k).$$

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\pi_*(I_{k-1}) \arrow[r] \arrow[d] & \pi_*(I_k) \arrow[r] \arrow[d] & \pi_*(I_{k+1}) \arrow[r, "d"] \arrow[d] & {} \\
\pi_*(I_{k-1}) \oplus \pi_*(J_{k-1}) \arrow[r] \arrow[d] & \pi_*(I_k) \oplus \pi_*(J_k) \arrow[r] \arrow[d, "(2)"'] & \pi_*(I_{k+1}) \oplus \pi_*(J_{k+1}) \arrow[d, "(3)"'] \arrow[u, dashed, "(4)"'] & (3) \arrow[d] \\
\pi_*(J_{k-1}) \arrow[r, "d"] \arrow[d] & \pi_*(J_k) \arrow[r, "d", "(1)"'] \arrow[d] & \pi_*(J_{k+1}) \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}
$$

taking cohomology we obtain a natural map δ

$$R^{k-1}\pi_* H \xrightarrow{\delta} (R^k\pi_*)(F) \to (R^k\pi_*)(G) \to \underbrace{(R^k\pi_*)(H)}_{\text{exactness is clear}} \xrightarrow{\delta} (R^{k+1}\pi_*)(F)$$

<u>Exercise</u> this is exact.

Construction of injective resolutions.

Last time exercise: Which abelian groups are injective?

<u>Answer</u> A is injective $\iff$ $\forall x \in A$ $\forall n \in \mathbb{Z}_{\geq 1}$ $\exists y$ s.t. $ny = x$.

for instance if K is a field of char 0, then K, and any vector space over k is injective.

Prop If I is injective, then $\forall \pi: X \to Y$ $\pi_* I$ is injective

proof Pick $0 \to F \to G \to H \to 0$ a SES

Consider $0 \to \mathrm{Hom}(H, \pi_* I) \to \mathrm{Hom}(G, \pi_* I) \to \mathrm{Hom}(F, \pi_* I) \to 0$
we need to see that this is exact.
applying adjunction this sequence is
$$0 \to \mathrm{Hom}(\pi^{-1}H, I) \to \mathrm{Hom}(\pi^{-1}G, I) \to \mathrm{Hom}(\pi^{-1}F, I) \to 0$$
since π^{-1} is exact $0 \to \pi^{-1}F \to \pi^{-1}G \to \pi^{-1}H \to 0$ is exact,
and since I is injective, the sequence in question is exact $\lrcorner$

Let X top space, let X^{discr} be the "discretization": as a set $|X^{\mathrm{discr}}| = |X|$, but with discrete topology.

Suppose for every point $x \in X^{\mathrm{discr}}$ we choose an abelian group A_x $\to$ obtain a sheaf on X^{discr} by $A(U) = \prod_{x \in U} A_x$ if A_x is injective $\forall x \in X$ then

A is injective.

By the proposition $\pi_* A$ is also injective.

A way to construct resolutions

F a sheaf on X $\to$ $\pi^{-1}F$ = a collection of abelian groups $(F_x)_{x \in X}$

if we are over a field these F_x are already injective, otherwise embed F_x into I_x, some injective group.

we obtain embedding $I(U) = \prod_{x \in U} I_x$
$$\pi^{-1}F \longrightarrow I$$
this induces a map $F \to \pi_* I$, which is injective
$(\mathrm{Hom}(\pi^{-1}F, I) = \mathrm{Hom}(F, \pi_* I))$ (Exercise Compute cohomology of that space with 4 points

with coefficients in the constant sheaf corresponding to k, where k is a field of char 0.