F

Direct, inverse image of sheaves $f: X \to Y$

Direct image was easy $(f_* F)(U) = F(f^{-1}(U))$ but inverse image had complicated 2-step definition.

But in the end we have seen $Hom(f^{-1}F, G) = Hom(F, f_* G)$, so $f^{-1}F$ represents a functor $G \to Hom(F, f_* G)$, so is described by $Hom(f^{-1}F, -)$.

automatically f^{-1} is right exact.

In fact it is exact. (both left-right exact, so preserves ker's coker's)

__Prop__ $f: X \to Y$ $g: Y \to Z$, then for $F \in Sh_Z$ we have $f^{-1} g^{-1} F \simeq (gf)^{-1} F$.

for any $G \in Sh_X$ $Hom(f^{-1}(g^{-1}(F)), G) \simeq Hom(g^{-1}(F), f_*(G)) = Hom(F, g_*(f_*(G)))$

$Hom((gf)^{-1}F, G) \cong Hom(F, (gf)_*(G))$

$(g_* f_*) G = g_*(f_*(G))$ implies

So $(gf)^{-1}F$ is isomorphic to $f^{-1}(g^{-1}(F))$ by Yoneda Lemma (i.e. they represent the same functor).

Consider special case i_p : point $\to p \in X$

What is $i_p^{-1}(F)$?

__Remark__ presheaves on the point are sheaves

a presheaf on a point is G(point) abelian group.

so sheafification does nothing over a point.

and $\quad i_p^{-1}(F)(\{\text{point}\}) = \varinjlim_{\substack{U \subset X \\ \cup \\ p}} F(U) = \bigsqcup F(U) / \sim$

$V \subset U$
$s \in F(U)$
$s \sim s|_V$

i.e. this is the stalk of F at p.

$$\boxed{\text{New notation} \quad s|_V = \mathrm{Res}_{V,U}(s)}$$

Prop $\quad i_p^*$ is exact

Proof take a left exact sequence $0 \to F \xrightarrow[f]{} G \xrightarrow[g]{} H$ $\quad (F = \ker(G \to H))$

consider

$$0 \to \underbrace{F_p}_{\boxed{i_p^* F}} \xrightarrow{f_p} G_p \xrightarrow{g_p} H_p$$

Notation $F_p = i_p^* F$

claim this is exact.

Suppose $s \in F_p$ such that $f_p(s) = 0$

$s \in F(U)$ $\quad f(s) \in G(U)$ $\quad f_p(s) = 0 \Rightarrow$

open $U \ni p$ $\quad \exists V \subset U$ open s.t. $f(s)|_V = 0$

$f(s)|_V = f(s|_V)$

since f is injective, this implies

$s|_V = 0 \Rightarrow s = 0$ in F_p.

$$(*) \quad 0 \to F \xrightarrow{f} G \xrightarrow{g} H$$

$$G \searrow \mathrm{Im}(g) = \mathrm{Coker}(f) \nearrow H$$

$\mathrm{Im}(g) \to H$ is injective $\Longleftarrow$ since $\mathrm{Coker}(f) = \mathrm{Coker}/\mathrm{Ker}\, g) = \underline{= \mathrm{Ker}}(\mathrm{Coker}(g))$

applying i_p^{-1} to $(*)$

$$0 \to F_p \xrightarrow{(1)} G_p \longrightarrow H_p$$

$$G_p \xrightarrow{(3)} (\mathrm{Im}(g))_p \xrightarrow{(2)} H_p$$

(1) is injective

(2) is injective $\quad$ since i_p^{-1} is right exact, we know that (3) is surjective.

Some exercises

$$\begin{tikzcd}
 & & & \mathrm{Im}(g_p) \arrow[dr, "(6)"] & \\
0 \arrow[r] & F_p \arrow[r, "i_p", "(1)"'] & G_p \arrow[ur, "(4)"] \arrow[rr, "g"] \arrow[dr, "(3)"'] & & H_p \\
 & & & (\mathrm{Im}\, g)_p \arrow[uu, "(5)"] \arrow[ur, "(2)"'] &
\end{tikzcd}$$

$(4) = \mathrm{Coker}\,(1) \Rightarrow \exists (5)$ making $(3), (4), (5)$ commutative

(4) is surjective $\Rightarrow (5)$ is surjective.

(3) surjective $\Rightarrow$ $(2), (5), (6)$ commutative

$\Rightarrow (5)$ is injective.

$x \in \mathrm{Ker}\, g \Rightarrow x \in \mathrm{Ker}\,(4) \Rightarrow x \in \mathrm{Ker}$ Mistake

$$\begin{tikzcd}
 & & & \mathrm{Coker}(i_p) \arrow[dr, "(6)"] \arrow[dd, "(5)"] & \\
0 \arrow[r] & F_p \arrow[r, "i_p", "(1)"'] & G_p \arrow[ur, "(4)"] \arrow[rr, "g"] \arrow[dr, "(3)"'] & & H_p \\
 & & & (\mathrm{Im}\, g)_p \arrow[ur, "(2)"'] &
\end{tikzcd}$$

$\exists (5)$ making it commutative

$(2), (6), (5)$ is comm.

$g, (4), 6$

If x is s.t. $g(x) = 0 \Rightarrow (2)(3)\, x = 0$ (2) is injective

$\Rightarrow (3)\, x = 0 \Rightarrow (5)(4)(x) = 0 \Rightarrow (2)(5)(4)\, x = 0$

$= (6)(4)\, x = 0 \Rightarrow (4)\, x = 0 \Rightarrow x \in \mathrm{Im}\, i_p.$

why?

Restart the proof.

Suppose $0 \to F \to G \to H \to 0$ is exact

$\Rightarrow$ by right exactness of i_p^{-1} $F_p \to G_p \to H_p \to 0$ is right exact

by what we proved $F_p \to G_p$ is injective

so $0 \to F_p \to G_p \to H_p \to 0$ is exact.

So i_p^{-1} sends short exact sequences to exact.

Prop suppose a functor sends short exact sequences to short exact sequences. Then it sends arbitrary exact sequences to exact.

$0 \to A \to B \to C$ exact

$0 \to F(A) \to F(B) \to F(C)$ exact ?

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[dr] & C \arrow[r, "h"] & \mathrm{Coker}\, g \arrow[r] & 0 \\
 & & & \mathrm{Im} f \arrow[u] \arrow[dr] & & \\
 & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

are short exact sequences

after applying F they remain s.e.s.

in particular $F(\mathrm{Coker}\, f) = \mathrm{Coker}\, F(f)$

and $F(\ker h) = \ker(F(h))$.

$0 \to F(A) \to F(B) \to F(C) \to F(D) \to 0$ is exact.

so it send kernels to kernels

cokers to cokers.

<u>Claim</u> Given a sequence of sheaves on X $F \to G \to H$, it is exact in G if and only if $F_p \to G_p \to H_p$ is exact <u>for any</u> $p \in X$.

<u>Proof</u> $\mathcal{H}_p(F \xrightarrow{f} G \xrightarrow{h} H) = \mathrm{Coker}(F \to \ker h)$

cohomology

applying i_p^{-1} we have $\mathcal{H}(F \to G \to H)_p = \mathcal{H}(F_p \to G_p \to H_p)$

so we need to show that $\forall\ K \in Sh_X$

$K_p = 0 \ \forall p \implies K = 0.$

Proof take $U \subset X$ $s \in K(U)$.
$\forall p$ $\exists V \ni p$ open $V \subset U$ s.t. $s|_V = 0$
(because $K_p = 0$)
$\Rightarrow \exists$ open covering V_λ of U s.t.
$s|_{V_\lambda} = 0 \Rightarrow s = 0$ by the first sheaf axiom. $\square$

More conceptually. Given X
consider $X^{discr} = X$ as a set with discrete topology.

then we have a continuous map
$\pi : X^{discr} \to X$.

$\pi^{-1} F = \prod F_p$ sheaf on X^{discr} = collection of abelian groups $\{F_p\}_{p \in X}$.

π^{-1} is exact
π^{-1} is faithful
$Hom(F, G) \to Hom(\pi^{-1}F, \pi^{-1}G)$ is injective

Prop if a functor F is exact + faithful $\Rightarrow$
$F \to G \to H$ is exact $\Leftrightarrow$ $\mathcal{F}(F) \to \mathcal{F}(G) \to \mathcal{F}(H)$ is exact.

Proof to show we need to check $\mathcal{F}(K) = 0 \Rightarrow K = 0$
proof consider $K \to 0 \to K$ $\in Hom(K, K)$
$K \xrightarrow{id} K$
if $\mathcal{F}(K) = 0 \Rightarrow \mathcal{F}(0) = \mathcal{F}(Id_K)$
$\Rightarrow 0 = Id_K$. so $K \to 0$ is an isomorphism. $\square$

Prop for any $f: X \to Y$
f^{-1} is exact

Proof

compose with $X_{discr} \xrightarrow{\pi} X \xrightarrow{f} Y$

Given $F \to G \to H$ sequence of sheaves on Y
we need to check $f^{-1}(F) \to f^{-1}(G) \to f^{-1}(H)$ is exact
since π^{-1} is exact, faithful it is sufficient to check
that $\pi^{-1}f^{-1}(F) \to \pi^{-1}f^{-1}(G) \to \pi^{-1}f^{-1}(H)$ is exact.
So for each $p \in X$ we need to check
$i_p^{-1}f^{-1}(F) \to i_p^{-1}f^{-1}(G) \to i_p^{-1}f^{-1}(H)$ is exact.

$$
\begin{array}{ccccc}
i_p^{-1}f^{-1}(F) & \to & i_p^{-1}f^{-1}(G) & \to & i_p^{-1}f^{-1}(H) \\
\| & & \| & & \| \\
(i_p')^{-1}(F) & \to & (i_p')^{-1}G & \to & (i_p')^{-1}(H)
\end{array}
$$

$p \xrightarrow{i_p} X \to Y$, $\quad i_p' = i_{f(p)}$

which is exact because $F \to G \to H$ is exact. $\square$

So far:

$$
\begin{cases}
f^{-1} \text{ is exact} \\
f_* \text{ is left exact.}
\end{cases}
$$

Def $M \in$ abelian category $\mathcal{A}$ is **injective** if $Hom(-, M)$ is right exact $\iff$ exact.

projective if $Hom(M, -)$ is left exact.

Cohomology of Sheaves (generalizes all other cohomologies)

Given object M we want to have an

Def injective resolution of M is $0 \to M \to I^0 \to I^1 \to I^2 \to \dots$
which is an exact sequence where I^k is injective.

projective resolution of M is an exact sequence

$$\cdots \to P_1 \to P_0 \to M \to 0$$

where P_n is projective.