

F

Direct, inverse image of sheaves $f: X \rightarrow Y$
 • Direct image was easy $(f_* F)(U) = F(f^{-1}(U))$
 but inverse image had complicated 2-step definition.

But note that we have seen
 $\text{Hom}(f^{-1}F, G) = \text{Hom}(F, f_* G)$, so
 $f^{-1}F$ represents a functor $G \rightarrow \text{Hom}(F, f_* G)$,
 so is described by $\text{Hom}(f^{-1}F, -)$.

automatically f^{-1} is right exact.

In fact it is exact. (both left-right exact,
 so preserves ker's, cokern's)

Prop $f: X \rightarrow Y$ $g: Y \rightarrow Z$, then for $F \in \text{Sh}_Z$
 we have $f^{-1}g^{-1}F \cong (gf)^{-1}F$.

for any $G \in \text{Sh}_X$ $\text{Hom}(f^{-1}(g^{-1}(F)), G) \cong$
 $\text{Hom}(g^{-1}(F), f_*(G)) = \text{Hom}(F, g_*(f_*(G)))$
 $\text{Hom}((gf)^{-1}F, G) \cong \text{Hom}(F, (gf)_*(G))$
 $(g_* f_*)G = g_*(f_*(G))$ implies

So $(gf)^{-1}F$ is isomorphic to $f^{-1}(g^{-1}(F))$ by
 Yoneda Lemma (i.e. they represent the same functor).

Consider special case $i_p: \text{point} \rightarrow X$

What is $i_p^{-1}(F)$?

Remark presheaves on the point are sheaves
 a presheaf on a point is $G(\text{point})$ a abelian group.
 so sheafification does nothing over a point.

and $i_p^{-1}(F) (\text{local}) = \lim_{\substack{U \subset X \\ p \in U}} F(U) = \varinjlim_{\substack{V \subset U \\ s \in F(U)}} F(U)$
 i.e. this is the stalk of F at p .

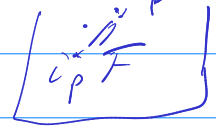
New notation
 $s|_V = \rho_{V,U}(s)$

Prop i_p^{-1} is exact

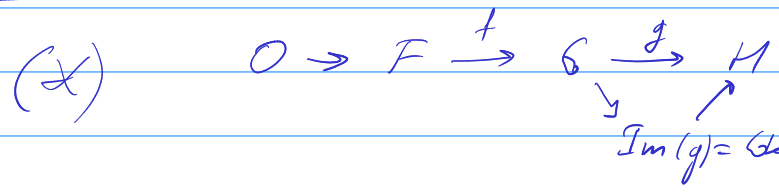
Proof take a left exact sequence $0 \rightarrow F \xrightarrow{f} G \rightarrow H$ ($F = \ker(G \rightarrow H)$)

consider $0 \rightarrow F_p \xrightarrow{f_p} G_p \rightarrow H_p$ claim this is exact.

Notation
 $F_p = i_p^{-1} F$

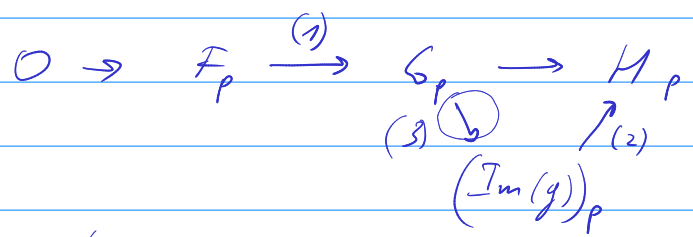


Suppose $s \in F_p$ such that $f_p(s) = 0$
 $s \in F(U)$ $f(s) \in G(U)$ $f_p(s) = 0 \Rightarrow$
 open $U \ni p$ $\exists V \subset U$ open s.t. $f(s)|_V = 0$
 since f is injective, this implies $f(s|_V) = 0$
 $s|_V = 0 \Rightarrow s = 0 \in F_p$



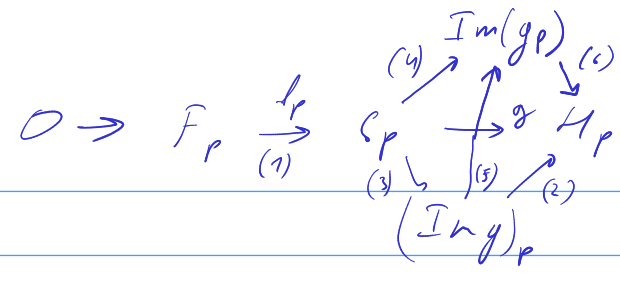
$\text{Im}(g) \rightarrow H$ is injective \Leftarrow since $\text{Gen}(f) = \text{Gen}(\text{Ker } g) = \text{Ker}(\text{Gen}(g))$

applying i_p^{-1} to (*)

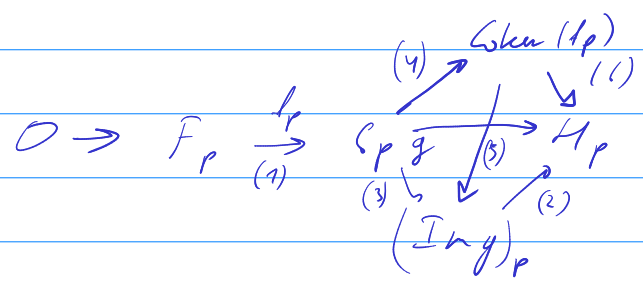


(1) is injective since i_p^{-1} is
 (2) is injective right exact, we know that (3) is
surjective.

Some exercises



(4) = $\text{Ker}(1) \Rightarrow \exists (5)$ making (3), (4), (5) commutative
 (4) is surjective \Rightarrow (5) is surjective.
 (3) surjective \Rightarrow (2), (5), (4) commutative
 \Rightarrow (5) is injective.
 $x \in \text{Ker } g \Rightarrow x \in \text{Ker}(4) \Rightarrow x \in \text{Ker}$ mistake



$\exists (3)$ making it commutative
 (2), (6), (5) is comm.
 $g, (4), 1$

If x is s.t. $g(x) = 0 \Rightarrow$ (2) (3) $x = 0$ (2) is injective
 \Rightarrow (3) $x = 0 \Rightarrow$ (5) (4) $(x) = 0 \Rightarrow$ (2) (5) (4) $x = 0$
 $=$ (6) (4) $x = 0 \Rightarrow$ (4) $x = 0 \Rightarrow x \in \text{Im } f_p$
 why?

Restate the proof.

Suppose $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$ is exact
 \Rightarrow by right exactness of i_p^{-1} $F_p \rightarrow G_p \rightarrow H_p \rightarrow 0$ is right exact

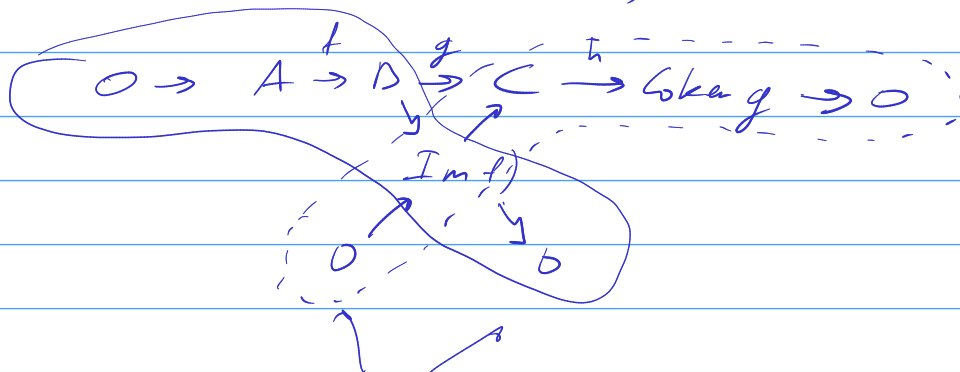
by what we proved $F_p \rightarrow G_p$ is injective
 so $0 \rightarrow F_p \rightarrow G_p \rightarrow H_p \rightarrow 0$ is exact.

So i_p^{-1} sends short exact sequences to exact.

Prop suppose a functor sends short exact sequences to short exact sequences. Then it sends arbitrary exact sequences to exact.

$$0 \rightarrow A \rightarrow B \rightarrow C \quad \text{exact}$$

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \quad \text{exact?}$$



one short exact sequences

after applying F they remain S.E.S.

in particular $F(\text{Coker } f) = \text{Coker } F(f)$

and $F(\text{Ker } h) = \text{Ker } (F(h))$

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow F(D) \rightarrow 0 \quad \text{is exact.}$$

so it send kernels to kernels
cokernels to cokernels.

Claim Given a sequence of spaces

on X $F \rightarrow G \rightarrow H$, it is exact iff
it and only if $F_p \rightarrow G_p \rightarrow H_p$ is exact
for any $p \in X$.

Proof $\mathcal{H}(F \rightarrow G \rightarrow H) = \text{Coker}(F \rightarrow \text{Ker } h)$
where \mathcal{H} is homology

applying i_p^{-1} we have $\mathcal{H}(F \rightarrow G \rightarrow H)_p = \mathcal{H}(F_p \rightarrow G_p \rightarrow H_p)$
so we need to show that $\forall K \in \mathcal{H}_x$
 $K_p = 0 \quad \forall p \Rightarrow K = 0$.

Proof take $U \subset X$ $s \in K(U)$.
 $\forall p \exists V \ni p$ open $V \subset U$ s.t. $s|_V = 0$
 (because $K_p = 0$)
 $\Rightarrow \exists$ ^{open} covering V_i of U s.t.
 $s|_{V_i} = 0 \Rightarrow s = 0$ by the first sheaf axiom. └

More conceptually. Given X
 consider $X^{\text{discr}} = X$ as a set
 with discrete topology.

then we have a continuous map
 $\pi: X^{\text{discr}} \rightarrow X$
 $\pi^{-1}F = \prod F_p$ sheaf on $X^{\text{discr}} =$
 collection of abelian groups
 $\{F_p\}_{p \in X}$.
 π^{-1} is exact
 π^{-1} is faithful.
 $\text{Hom}(F, G) \rightarrow \text{Hom}(\pi^{-1}F, \pi^{-1}G)$ is injective

Prop if a functor F is exact + faithful \Rightarrow
 $F \rightarrow G \rightarrow H$ is exact $\Leftrightarrow F(F) \rightarrow F(G) \rightarrow F(H)$ is exact.

Proof to show we need to check $F(K) = 0 \Rightarrow K = 0$
proof consider $K \rightarrow 0 \rightarrow K$ $\in \text{Hom}(K, K)$
 $K \xrightarrow{\text{id}} K$
 if $F(K) = 0 \Rightarrow F(0) = F(\text{Id}_K)$
 $\Rightarrow 0 = \text{Id}_K$. so $K \rightarrow 0$ is an isomorphism. └

Prop for any $f: X \rightarrow Y$
 f^{-1} is exact

Proof

compose with $X \xrightarrow{\pi} X \xrightarrow{f} Y$

Given $F \rightarrow G \rightarrow H$ sequence of sheaves on Y
 we need to check $f^{-1}(F) \rightarrow f^{-1}(G) \rightarrow f^{-1}(H)$ is exact
 since π^{-1} is exact, faithful it is sufficient to check
 that $\pi^{-1}f^{-1}(F) \rightarrow \pi^{-1}f^{-1}(G) \rightarrow \pi^{-1}f^{-1}(H)$ is exact.

So for each $p \in X$ we need to check

$i_p^{-1}f^{-1}(F) \rightarrow i_p^{-1}f^{-1}(G) \rightarrow i_p^{-1}f^{-1}(H)$ is exact.

$(i_p^{-1})^{-1}(F) \rightarrow (i_p^{-1})^{-1}(G) \rightarrow (i_p^{-1})^{-1}(H)$ $p \xrightarrow{i_p} X \rightarrow Y$
 $i_p^{-1} = f^{-1}(p)$

which is exact because $F \rightarrow G \rightarrow H$ is exact. \square

So for:

f^{-1} is exact
 f_* is left exact.

Def $M \in$ abelian category \mathcal{A} is
injective if $\text{Hom}(-, M)$ is right exact
 \Leftrightarrow exact.

projective if $\text{Hom}(M, -)$ is left exact.

Homology of sheaves (generalizes all other cohomologies)

Given object M we want to have an

Def injective resolution of M is $0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots$
 which is an exact sequence where I^k is injective.

Projective resolution of M is an exact sequence

$$\dots P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

where P_i is projective.