Teaching

Mo: 9:45 - 11:15

Fr 11:30 - 13:00

Today: 1) More on plane curves
2) Approach to schemes via functor of points
3) Fiber products, products

Next Morphisms of schemes, properties.

1) Rational parametrization of conics

quadratic curves in $\mathbb{P}^2$

Again Q = quadratic form in x, y, z over alg. closed field can be transformed to

$x^2 + y^2 - z^2$

$x^2 + (y-z)(y+z)$

change of variables

$x^2 + yz$

Sometimes it is better to make quadratic forms not $\sum_{i=1}^{n} X_i^2$, but

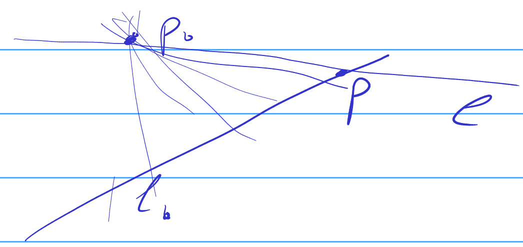
$X_1^2 + X_2X_3 + X_4X_5 + \dots + X_{2k}X_{2k+1}$ (n odd)

$X_1X_2 + X_3X_4 + \dots$ (n even).

Given a conic, a point on a conic, and a line not containing the point

i.e. on $x^2 + yz = 0$ $\quad (0,1,0) = p_0$

line $\{(s,0,t) \mid (s,t) \neq (0,0)\} = \ell_0$ $\quad \mathbb{P}^1 \subset \mathbb{P}^2$



Let $p = (s,0,t) \in \ell_0$

Then ℓ = span of (p_0, p), i.e.

ℓ is parametrized by

$p_0 \cdot u + p \cdot v \qquad \{(vs, u, vt) \mid (u,v) \neq (0,0)\} = \ell$

$\ell \cap Q$: $(vs)^2 + u \cdot vt = 0$, solve for u, v.

$v(vs^2 + ut) = 0$ $v=0$ is always a solution

corresponds to $(0,u,0)$ or $(0,1,0)$ $= p_0$

$vs^2 + ut$ gives another intersection point of $Q \cap \ell$

$u = s^2$, $v = -t$

plug in, $(vs, u, vt) = (-st, s^2, -t^2)$.

So we obtained a map $\mathbb{P}^1 \to \mathbb{P}^2$ given by sending (s,t) to $(-st, s^2, -t^2)$. This is a bijection on Q. (To see this, we can cover Q by open sets, on each set define an inverse.

Q $(x^2 + yz = 0)$ $x \neq 0 = \underline{U_x}$ $\frac{y}{x} = \alpha$ $\frac{z}{x} = \beta$

equation $\alpha\beta + 1 = 0$

$\cdot p_0$

α, β $\ell = u(1, \alpha, \beta) + v(0,1,0)$ $\ell \cap \ell_0$

$(u, u\alpha + v, u\beta)$ $v = -u\alpha$

$(u, 0, u\beta) = p$

on U_x $(\alpha, \beta) \to (1, \beta) \in \mathbb{P}^1$

on $\underline{U_y}$ $\frac{x}{y} = \gamma$ $\frac{z}{y} = \delta$ equation $\gamma^2 + \delta = 0$

compute the map on U_y

$\ell = u(\gamma, 1, \delta) + v(0,1,0)$ $\ell \cap \ell_0$: $v = -u$

$U_y \ni (\gamma, \delta) \to (\gamma, \delta) \in \mathbb{P}^1$

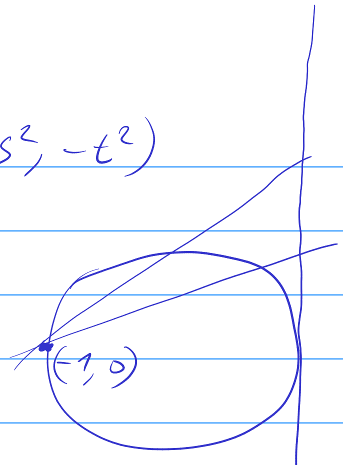
on U_z ε, ζ eq. $\varepsilon^2 + \zeta = 0$

$\varepsilon = x/z$, $\zeta = y/z$

$\ell = u(\varepsilon, \zeta, 1) + v(0,1,0)$ $(\varepsilon, 1)$

$\varepsilon, \zeta \in U_z$ go to $(\varepsilon, 1) \in \mathbb{P}^1$

Look again $(s,t) \to (-s.t, s^2, -t^2)$



<u>More traditional</u>

parametrize circle by points on line

$x^2+y^2=1$ $\quad \left(\frac{2s}{s^2+1}\right)^2 + \left(\frac{s^2-1}{s^2+1}\right)^2 = 1$ ← the same parametrization in different coordinates

so $s \mapsto \left(\frac{2s}{s^2+1}, \frac{s^2-1}{s^2+1}\right)$ parametrizes the circle.

used to classify pythagorean triples
(good example how alg. geometry helps number theory)

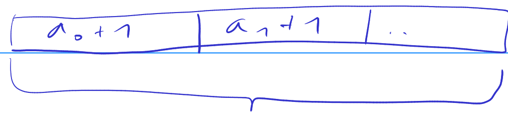
<u>Remark</u> changing coordinates we can consider $(s,t) \to (s^2, st, t^2)$. This parametrizes
$\qquad\qquad x \quad y \quad z \qquad\qquad xz = y^2$.

This is example of Veronese embedding.
More generally, take $\mathbb{P}^k$, coordinates $(X_0, .., X_k)$.
list all monomials of degree n. There are

gives a map $\binom{n+k}{k} - 1 = N$
$\mathbb{P}^k \to \mathbb{P}^{\binom{n+k}{k}-1}$

$X_0^{a_0} \dots X_k^{a_k}$

$\updownarrow$



$n+k+1$

This is an embedding
(so $\mathbb{P}^k$ is realized as a closed subscheme of $\mathbb{P}^N$).

there are $n+k$ positions to put k vertical lines
so $\# = \binom{n+k}{k}$.

More generally
If R is a graded ring, Let $n \in \mathbb{Z}_{>0}$, Let
$R^{(n)} = \bigoplus_{k=0}^{\infty} R_{nk}$. Then $\mathrm{Proj}(R^{(n)}) \cong \mathrm{Proj}(R)$.
Why? Let U_f be a basic open set of $\mathrm{Proj}(R)$

for f of degree k. Then $f^n \in R^{(n)}_k$

clearly $\left[(f^n)^{-1} R^{(n)}\right]_0 = \left[f^{-1} R\right]_0$ so $U_f \simeq \boxed{U^{R^{(n)}}_{f^n}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = \left[(f^n)^{-1} R\right]_0$

corresponding basic open of $\mathrm{Proj}\ R^{(n)}$.

So for $R = k[X_0, \dots, X_k]$ $R^{(n)}$ is generated by all monomials of degree n, so it is a quotient of the polynomial ring in $N+1$ variables by some ideal. So $\mathbb{P}^k$ is isomorphic to a closed subscheme of $\mathbb{P}^N$.

Terminology: a scheme is <u>projective</u> if it is a closed subscheme of a projective space (equivalently, it is Proj (quotient of $k[X_0, \dots, X_n]$), equivalently it is Proj (finitely generated graded ring satisfying $R_0 = k$).

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<u>Ex</u> weighted proj space $\subset$ projective space

$\mathrm{Proj}\ k[X_0, \dots, X_n]$

$\deg X_i = d_i$

Let $L = \mathrm{l.c.m}(d_0, \dots, d_n)$, list all monomials of degree L. If there are N of them obtain embedding into $\mathbb{P}^{N-1}$.

Maybe L doesn't work, but some multiple of L does.

(want every monomial of degree kL be divisible by some monomial of degree L). good exercise...

Functor of points.

Recall if we have a system of equations, we should study solutions over arbitrary rings, Yoneda lemma guarantees that we don't lose information:

$$\text{Rings} \longrightarrow \text{Fun}(\text{Rings}, \text{Sets})$$

the category of rings

encodes the coefficient ring

set of solutions

encodes system of equations

$$\begin{array}{ccc} \text{Rings} & \longrightarrow & \text{Fun}(\text{Rings}, \text{Sets}) \\ \Psi & & \Psi \\ R & \longrightarrow & \left(\mathbb{k} \to \text{Hom}_{\text{rings}}(R, \mathbb{k}) \right) \end{array}$$

a functor

This is a contravariant functor.

(if $\varphi: R_1 \to R_2$) then $\forall$ $\mathbb{k}$ we have a map of sets

$$\text{Hom}(R_2, \mathbb{k}) \to \text{Hom}(R_1, \mathbb{k}).$$

for each R the functor $\mathbb{k} \to \text{Hom}(R, \mathbb{k})$ is called the functor of points.

We don't loose information means that by Yoneda lemma the category of rings embeds into the category of functors Rings $\to$ Sets.

embeds := bijection on Homs

injection : faithful

surjection : full

What about schemes?

We have a category of schemes: Objects are schemes, morphisms are morphisms of schemes.
Denoted by Sch.

We have a full subcategory Aff Sch of affine schemes.
Objects of Aff Sch are schemes of the form $\mathrm{Spec}(R)$ for rings R.
We know that $\forall\, R_1, R_2$

$$\boxed{\mathrm{Mor}(\mathrm{Spec}(R_1), \mathrm{Spec}(R_2)) = \mathrm{Hom}(R_2, R_1).}$$

So Aff Sch is isomorphic to Rings^{op}

morphisms go in the opposite direction.

The functor of points can be interpreted as

$$\begin{array}{ccc} \text{Aff Sch} & \longrightarrow & (\text{Rings} \to \text{Sets}) \\ \Psi & & \Psi \\ \mathrm{Spec}(R) & & \mathbb{k} \to \mathrm{Hom}(R, \mathbb{k}) \\ X & \longrightarrow & (\mathbb{k} \to \mathrm{Mor}(\mathrm{Spec}(\mathbb{k}), X)) \end{array}$$

We generalize this to arbitrary schemes by the same formula

$$\begin{array}{ccc} \text{Sch} & \longrightarrow & (\text{Rings} \to \text{Sets}) \\ X & \longrightarrow & (\mathbb{k} \xrightarrow[F_X]{} \mathrm{Mor}(\mathrm{Spec}(\mathbb{k}), X)) \end{array}$$

Variation of this Schemes over a field, instead of Mor take Mor over that field.

Theorem This functor is an embedding of Sch into $\mathrm{Fun}(\text{Rings}, \text{Sets})$.

Sketch of proof

We need to understand what are morphisms between two arbitrary schemes. X_1, X_2 in terms of morphisms between affine schemes.

X_1 is a union of open affine schemes $\{U_\lambda\}_{\lambda\in\Lambda}$

$$\mathrm{Mor}(X_1, X_2) = \left\{ \underbrace{(f_\lambda : U_\lambda \to X_2)_{\lambda\in\Lambda}} \;\middle|\; \forall \lambda,\mu\in\Lambda \quad \underbrace{f_\lambda|_{U_\lambda\cap U_\mu} = f_\mu|_{U_\lambda\cap U_\mu}} \right\}$$

Let $U_\lambda = \mathrm{Spec}\, R_\lambda$

$$f_\lambda \in F_{X_2}(R_\lambda)$$

$U_\lambda \cap U_\mu$ does not have to be affine, however can be covered by affines

Exercise intersection of two open affines doesn't have to be affine.

$\underbrace{\mathbb{A}^2} \cup_U \underbrace{\mathbb{A}^2}$ $\quad U = \mathbb{A}^2 \setminus \{(0,0)\}$

Let's rewrite

$$f_\lambda|_{U_\lambda\cap U_\mu} = f_\mu|_{U_\lambda\cap U_\mu}$$

as follows: for any affine scheme V and any commutative diagram

$$\begin{array}{ccc} V & \to & U_\lambda \\ \downarrow & & \downarrow \\ U_\mu & \to & X_1 \end{array}$$

we want

$$\begin{array}{ccc} V & \to & U_\lambda \\ \downarrow & & \downarrow f_\lambda \\ U_\mu & \xrightarrow{f_\mu} & X_2 \end{array}$$

to be commutative

$\forall$ ring R and homomorphisms $\alpha: R_\lambda \to R$, $\beta: R_\mu \to R$ whenever

such that $\underbrace{\alpha(\underset{\underset{F_{X_1}(R_\lambda)}{\wedge}}{U_\lambda \to X_1})}_{\underset{F_{X_1}(R)}{\wedge}} = \beta(U_\mu \to X_1)$, also

$\alpha(f_\lambda) = \beta(f_\mu)$ holds.

$$\mathrm{Mor}(X_1, X_2) \longrightarrow \mathrm{Mor}(F_{X_1}, F_{X_2}) \ni \varphi$$

1) Why injective? Because each map $X_1 \to X_2$ is uniquely determined by the collection $\{f_\lambda\}$, which is nothing but the value of φ on $U_\lambda \to X_1$, with $U_\lambda \to X_1 \in F_{X_1}(R_\lambda)$

$$f_\lambda = \varphi(U_\lambda \to X_1)$$

2) Why surjective? Because given a collection of f_λ, it defines a morphism $X_1 \to X_2$ when some conditions are satisfied.

$$\begin{tikzcd} R_\lambda \arrow[dr] & \\ & R \\ R_\mu \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} F_{X_1}(R_\lambda) \arrow[r, "\varphi"] \arrow[d] & F_{X_2}(R_\lambda) \arrow[d] \\ F_{X_1}(R) \arrow[r, "\varphi"] & F_{X_2}(R) \\ F_{X_1}(R_\mu) \arrow[r, "\varphi"] \arrow[u] & F_{X_2}(R_\mu) \arrow[u] \end{tikzcd}$$

Commutative diagram because φ is a natural transformation.

The condition is clear from the commutativity.

Theorem is proved.

Remark In categorical language X_1 is a colimit of all of its open affine subschemes. Basically we used that morphisms $\mathrm{Hom}(-, X_2)$ commutes with colimits.

When $F = F_X$ is given we say X represents the functor F.

So we have a way to construct schemes by

1) Construct a functor
2) Prove that the functor is representable.

Examples . Fiber product

Setup X_1, X_2, Y schemes with morphisms

$$\begin{array}{ccc} & & X_2 \\ & & \downarrow \pi_2 \\ X_1 & \xrightarrow[\pi_1]{} & Y \end{array}$$

. Consider the functor

$$F(R) = \{ f_1 \in F_{X_1}(R),\ f_2 \in F_{X_2}(R) \mid \underset{\substack{\cap \\ F_Y(R)}}{\pi_1(f_1) = \pi_2(f_2)} \}$$

Theorem F is representable by a scheme called the fiber product. $X_1 \times_Y X_2$.

Example $Y = \operatorname{Spec} \mathbb{Z}$ (Spec k if we are over a field k)
Then this is called the product
for any scheme X there is a unique map to Y, so $\pi_1(f_1) = \pi_2(f_2)$ is automatic.

Example $\mathbb{A}^k \times \mathbb{A}^n = \mathbb{A}^{k+n}$
$\forall$ ring R $F_{\mathbb{A}^k}(R) = \underbrace{R \times \ldots \times R}_{k}$

Example $G_m = \mathbb{A}^1 \setminus \{0\} = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ ($= \operatorname{Spec} k[t, t^{-1}]$ if we are over k)

$F_{G_m}(R) = \{ x \in R \mid x^{-1} \text{ exists} \} = R^\times$.
Note that $R^\times$ is a commutative group, so in particular we have a map $R^\times \times R^\times \to R^\times$ (multiplication)

So automatically we obtain a morphism of schemes
$G_m \times G_m \to G_m$ (explicitly this corresponds to a homomorphism of rings $k[t, t^{-1}] \to k[u, u^{-1}, v, v^{-1}]$.
$t \to uv$

G_m is called a group scheme

($\forall R$ $F_X(R)$ is a group).

Proof of Theorem

$$\begin{array}{ccc} & & X_2 \\ & & \downarrow \pi_2 \\ X_1 & \xrightarrow{\pi_1} & Y \end{array}$$

Reduction 1 assume $U \subset Y$ $V_1 \subset \pi_1^{-1}(U)$

$V_2 \subset \pi_2^{-1}(U)$ are affine open.

$U = Spec(R)$ $V_1 = Spec(R_1)$ $V_2 = Spec(R_2)$

What is $V_1 \times_U V_2$? $= Spec(\underbrace{R_1 \otimes_R R_2})$

given rings R_1, R_2, homomorphisms

$f_i : R \to R_i$

$R_1 \otimes_R R_2 =$ generated by symbols $x \otimes 1$, $1 \otimes y$

$x \in R_1$, $y \in R_2$, mod relations

$(x_1 \otimes 1)(x_2 \otimes 1) = x_1 x_2 \otimes 1$ $\quad x_1 \otimes 1 + x_2 \otimes 1 = (x_1 + x_2) \otimes 1$

similarly for $1 \otimes y_1$, $1 \otimes y_2$

$f_1(z) \otimes 1 = 1 \otimes f_2(z)$ $\forall z \in R$.

(= tensor product as abelian groups).

because

$\underline{x \otimes y} := (x \otimes 1) \cdot (1 \otimes y)$ additively generate $R_1 \otimes_R R_2$.

Example $k[x_1, \dots, x_n] \otimes_k k[y_1, \dots, y_m] = k[x_1, \dots, x_n, y_1, \dots, y_m]$.

$$R/I \otimes R' = (R \otimes R') / (I \otimes 1)$$

In general $X_1 \times_Y X_2$ is obtained by gluing the

$U_1 \times_V U_2$.

Question is $\mathbb{P}^n \times \mathbb{P}^m$ projective?