R local ring, domain, $\varphi: R \to F(R)$. $F(R)$ is a field, hence local ring (max. ideal is $\{0\}$). $\varphi^{-1}(\{0\}) = \{0\}$ not maximal!

$\operatorname{Spec} F(R) \to \operatorname{Spec} R$ The map of the corresponding local rings is not this map!

x_0 goes to y_0

ideal $\{0\}$ ideal $\{0\}$

$\mathcal{O}_{\operatorname{Spec} R, y_0} = R_{(0)} = F(R)$, so the induced map of local rings is $F(R) \to F(R)$ identity, so it is a local ring map.

Maybe for schemes morphisms of ringed spaces is the same as morphisms of locally ringed spaces? (generally the condition on locally ringed spaces is more strict).

Alternative point of view on points (from Mathoverflow post)

Let R be a ring.

<u>New definition</u> a point on $\operatorname{Spec}(R)$ is a pair $x = (k_x, ev_x)$, where k_x is a field, $ev_x: R \to k_x$ is a ring homomorphism.

<u>Ex</u> $R = k[x_1, \dots, x_n]/(f_1, \dots, f_n)$, then a point is a field extension of k, and a solution to the system of equations $(f_1, \dots, f_n) = 0$.

Elements of R are called functions. For each $f \in R$, each point $x = (k_x, ev_x)$ the value $f(x) = ev_x(f) \in k_x$.

Topology: a set Z is closed if $\exists$ collection of functions $\{f_\lambda\}$ s.t. $x \in Z \iff f_\lambda(x) = 0 \ \forall \lambda$.

$\forall f \in R$ the set $U_f = \{x \in \operatorname{Spec} R \mid f(x) \neq 0\}$ is open. (called a basic open or principal open)

Each open set is a union of such, so basic opens form a basis of topology.

Everything works out. (we can define schemes, glue them, ...).

One difference:

<u>Def</u> A topological space is T_0 (or Kolmogorov) if $\forall x, y, x \neq y$ we have $\exists U \ni x$ open nbh s.t. $y \notin U$ or $\exists U \ni y$ open nbh. s.t. $x \notin U$.

<u>Remark</u> if X is not T_0, suppose $x \neq y$ s.t. the condition fails. Then for each open set $U \subset X$ $x \in U \iff y \in U$. So we say x and y are indistinguishable.

The relation "indistinguishable" is an equivalence relation clearly.

$X/_{\text{indistinguishable}}$ is always a T_0 space.

It turns out $\text{New Spec}(R)/_{\text{indist.}} = \text{Spec}(R)$.

Proof 1) $\longrightarrow$ given $x = (k_x, ev_x) \in \text{New Spec}(R)$ take $\ker ev_x \subset R$ is a prime ideal, so defines a point on $\text{Spec}(R)$

2) Surjective $\forall p \subset R$ prime take $x_p = \left(F(R/p), ev_x \text{ is the map } R \to R/p \to F(R/p)\right)$, $F(R/p) = R_p/m_p$

It remains to show that

x, y are indistinguishable $\iff \ker ev_x = \ker ev_y$.

$\Updownarrow$

$\forall f \quad x \in U_f \iff y \in U_f$

$\Updownarrow$

$x \notin U_f \iff y \in U_f$

$\Updownarrow$

$ev_x(f) = 0 \quad ev_y(f) = 0$

$\Updownarrow$

$\ker(ev_x) = \ker(ev_y)$. $\square$

The topologies match:

New Spec may look more intuitive.

Or we could define $\text{Spec}(R)$ by $\text{New Spec}(R)/\sim$, $\sim$ indistinguishable.

Prop

Any scheme is T_0.

Proof $x, y \in X$ scheme $x \neq y$:

Let $U \ni x$ be affine neighborhood of x.

If $y \notin U$ we are done.

otherwise $y \in U$, since $U = \text{Spec}\, R$ the statement follows from the one for $\text{Spec}\, R$ $\square$

Ex $x = (k_x, ev_x)$, take any extension of k_x, $k_x \subset L$

Let $y = (L, \text{inc} \circ ev_x)$. Then $x \sim y$.

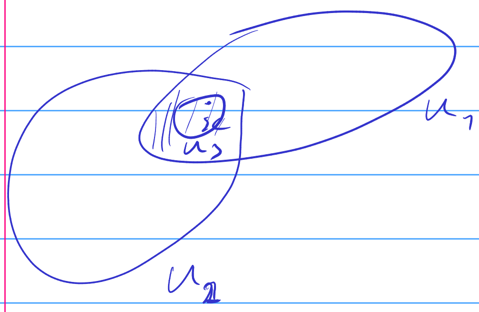
inclusion map

Remark $x \sim y$ if $\forall f \in R$ $f(x) = 0 \iff f(y) = 0$.

Break until 10:38

Let X be a scheme. Let k be a field, Let's understand morphisms $\text{Spec}\, k \to X$. Note: if X is affine, say $X = \text{Spec}\, R$, then $\text{Mor}(\text{Spec}\, k, X) = \text{Hom}(R, k)$.

Let $\varphi: \operatorname{Spec} k \to X$. $\operatorname{Spec} k = 1$ point, so $\varphi(\operatorname{Spec} k)$ is some point of X, call it x. φ continuous means nothing. φ is a map of ringed spaces means $\forall\, U \subset X$ open we have a homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_{\operatorname{Spec} k}(\varphi^{-1} U)$. If $U \not\ni x$ this is empty information ($\varphi^{-1} U = \emptyset$). If $U \ni x \Rightarrow \varphi^{-1} U = \operatorname{Spec} k$, the data is a homomorphism $\mathcal{O}_X(U) \to k$.



$$\begin{tikzcd} \mathcal{O}_X(U_1) \arrow[rr] \arrow[dr] & & k \\ & \mathcal{O}_X(U_3) \arrow[ur] & \\ \mathcal{O}_X(U_2) \arrow[ur] \arrow[uurr] & & \end{tikzcd}$$

these are commutative diagrams by the condition on map of ringed spaces.

So the data is equivalent to a homomorphism $\mathcal{O}_{X,x} \to k$.

So in the definition of morphism of schemes we need an extra condition: $\begin{matrix} X & \to & Y \\ \Psi & & \Psi \\ x & \to & y \end{matrix}$ induces $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, we assume the preimage of the maximal ideal in $\mathcal{O}_{X,x}$ is the maximal ideal in $\mathcal{O}_{Y,y}$.

so the map $\mathcal{O}_{X,x} \to k$ is such that it factors as

$$\mathcal{O}_{X,x} \to \underbrace{\mathcal{O}_{X,x}/m}_{\text{is called the residue field of } x \in X} \to k.$$

$k[[x]] \to k((x))$

power series $c_0 + c_1 x + \dots$

Laurent series $c_{-3} x^{-3} + c_{-2} x^{-2} + \dots$ a field.

We obtained: a map $\operatorname{Spec} k \to X$ is the same as 1) point $x \in X$ 2) homomorphism $k_x \to k$.

$k_x = \mathcal{O}_{X,x}/m$

We can consider $\widetilde{X} = \{ \text{pairs } (k, \varphi) \mid k \text{ is a field}, \varphi: \operatorname{Spec} k \to X \}$.

Then $X = \widetilde{X}/\sim$ where $(k_1, \varphi_1) \sim (k_2, \varphi_2)$ if $\operatorname{Im} \varphi_1 = \operatorname{Im} \varphi_2$.

equivalently, $\operatorname{Im}(\varphi_1) \subset U \iff \operatorname{Im}(\varphi_2) \subset U$ for all open U.

So we obtain analogous statement. So we could have said points of X are morphisms $\mathrm{Spec}\, k \to X$ for all fields k. These are called k-points.

Let $x, y \in X$, $x \neq y$. Sometimes $\exists\, U \ni x$, $U \not\ni y$ and $\exists\, U' \ni y$, $U' \not\ni x$. $\iff$ $\overline{\{y\}} \not\ni x$, $\overline{\{x\}} \not\ni y$.

Sometimes we have $x \in \overline{\{y\}}$ (then $y \notin \overline{\{x\}}$). Then we say that x is a specialization of y, and y is a generalization of x.

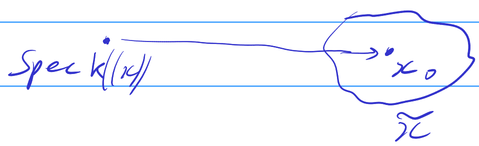
Example $X = \mathrm{Spec}(k[x]) = \mathbb{A}^1$, $x = 1 \in \mathbb{A}^1$ (corresponds to the ideal $(x-1)$) corresponds to the homomorphism $k[x] \to k$, $x \mapsto 1$. denote this by $\mathbb{1}$.

Another point ideal (0), homomorphism $k[x] \to k(x)$. denote by g. called "generic point".

$\forall f \in R$ $f(g) = 0 \Rightarrow f(1) = 0$, so $\bar{g} \ni \mathbb{1}$. So 1 is a specialization of g.

$R = k[[x]]$ x_0 special point $\tilde{x}$ generic

$\mathrm{Spec}\, k \to \mathrm{Spec}$



$R = k[x]$ $L = k(x)$

$R \to L$ $\mathrm{Spec}\, L \to \mathrm{Spec}\, R$

Send $\mathrm{Spec}\, L$ to $0 \in \mathrm{Spec}\, R$ instead of the generic point.

On stalks we have $k[x]_0 \longrightarrow k(x)$. not local!

invert all $p(x)$ s.t. $p(0) \neq 0$

From the "New Spec" viewpoint:

$\mathrm{Spec}\, L$ has residue field $k(x)$, but 0 has residue field k.

Generally $\quad x \in X \xrightarrow{f} y \in Y$ $\quad$ you want functions on $Y \to$ functions on X, and if f is a function on Y, the value of f at y should be equal to the value of $f^* f$ at x.

$$\begin{array}{ccc} O(Y) & \to & O(X) \\ \downarrow ev_y & & \downarrow ev_x \\ k_y & \dashrightarrow & k_x \end{array}$$

If we think $X =$ pairs (k_x, ev_x) $\quad Y =$ pairs $\ldots$, then $f: X \to Y$ must preserve k_x

Construction of a map $\operatorname{Spec} R \to \mathbb{P}^1$, where $\operatorname{Spec} R$ is connected, irreducible.

$\mathbb{P}^1 =$ set of all 1-dimensional linear subspaces in $\mathbb{A}^2$.

Consider 2×2 matrices M s.t. $M^2 = M$, rank $M = 1$.

We need to translate these to equations on

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad M^2 = M$$

b*c + d^2 - d, a + d - 1

$a+d=1 \Rightarrow$ eliminate a.

$$M = \begin{pmatrix} 1-d & b \\ c & d \end{pmatrix} \qquad R = k[b,c,d] / \underline{bc + d^2 - d = 0} \qquad X = \operatorname{Spec} R$$

this is the space of matrices with $\det = 0$ and $\operatorname{trace} = 1$.

$M \to \mathbb{P}^1$? $M \to$ Image of M

if $b \neq 0$ or $d \neq 0$ M goes to (b, d),

otherwise $b = d = 0$ $M = \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}$ goes to $(1, c)$.

$\mathbb{P}^1 = U_1 \cup U_2$ (x, y), $U_1: y \neq 0$

$U_2: x \neq 0$

$y \neq 0$: either $c \neq 0$ or $d \neq 0$.

$$U_c \cup U_d \to U_1$$

on U_c $\frac{1-d}{c}$, on U_d: $\frac{b}{d}$. On $U_c \cap U_d$ we have

$$\frac{1-d}{c} - \frac{b}{d} = \frac{d(1-d) - bc}{cd} = 0$$ so this is a well defined

function on $U_c \cup U_d$, obtain a map $U_c \cup U_d \to U_1$

Similarly define a function on U_b by $\frac{d}{b}$, on U_{1-d} by $\frac{c}{1-d}$.

on $U_b \cap U_{1-d}$ the difference is $\frac{d}{b} - \frac{c}{1-d} = \frac{d(1-d) - bc}{b(1-d)} = 0$.

So we have a well-defined function on

$U_b \cup U_{1-d}$, therefore a morphism $U_b \cup U_{1-d} \to \mathbb{A}^1 = U_2$.

On $(U_c \cup U_d) \cap (U_b \cup U_{1-d}) = (U_b \cap U_c) \cup (U_d \cap U_b) \cup (U_c \cap U_{1-d})$

$\cup (U_d \cap U_{1-d})$:

on $U_b \cap U_c$ our functions are $\frac{1-d}{c}, \frac{d}{b}$. We have

$(1-d)d = bc$ so $b \neq 0, c \neq 0 \Rightarrow 1-d \neq 0, d \neq 0$, so both functions

are mapped to $U_1 \cap U_2$ and coincide $\left(\frac{1-d}{c} \cdot \frac{d}{b} = 1\right)$.

Similarly we should be able to check that we obtain a map $X \to \mathbb{P}^1$

Going back to Proj

Recall $R = \bigoplus_{n=0}^{\infty} R_n$ graded ring.

points of $\mathrm{Proj}\, R = X$ are graded prime ideals p of R, such that $p \not\supset R_{>0}$. Basic open sets $f \in R_k$, U_f = prime ideals not containing f. $\mathcal{O}_X(U_f) = (R[f^{-1}])_0$.

Consider a family $\{U_{f_\lambda}\}$. When do we have $\bigcup_\lambda U_{f_\lambda} = X$?

<u>Ideals $\Leftrightarrow$ closed sets correspondence:</u>

If $I \subset R$ is a graded ideal, then $Z(I) = \{p \in \mathrm{Proj}(R) / p \supset I\}$

If $Z \subset \mathrm{Proj}(R)$ is a set, then $V(Z) = \{f \mid f \in p, \text{ all } p \in Z\}$.

Note that Z is closed $\Leftrightarrow Z = Z(I)$, some I.

Clearly $I_1 \subset I_2 \Rightarrow Z(I_1) \supset Z(I_2)$,

$Z_1 \subset Z_2 \Rightarrow V(Z_1) \supseteq V(Z_2)$.

$V(Z(I)) \supset I \qquad Z(V(A)) \supset A \Rightarrow Z(V(Z(I))) \subseteq Z(I)$

So if $A = Z(I)$, i.e. A is closed $\Rightarrow Z(V(A)) = A$.

similarly, if $I = V(A)$, some A then $V(Z(I)) = I$.

For Spec we had $V(Z(I)) = \mathrm{rad}(I)$.

<u>Question</u> What is it for Proj ?

$$V(Z(I)) = \bigcap_{\substack{p \supset I \\ p \text{ graded} \\ p \not\supset R_{>0}}} p.$$

Pass to R/I (again a graded ring)

$$\mathrm{Nil}(R) \subset \bigcap_{p \in \mathrm{Proj}\, R} p$$

Conversely, if x is not nilpotent, take p a maximal ideal which doesn't contain powers of x, then it is 1) prime.

2) $p \not\ni x$ so $p \in \mathrm{Proj}\, R$.

Let R be graded.

We can consider 2 things:

1) $\mathrm{Nil}(R) = \{x \in R \mid x^n = 0, \text{ some } n\}$

2) $\{x \in R \mid x \cdot R_{>n} = 0, \text{ some } n\}$

$= \bigcup_n (0 : R_{>n}) =: (0 : R_\infty)$

Assume R_0 is a domain. then if $x \in R_k$ $(k>0)$ $x \cdot R_{>n} = 0$ some n

$\Rightarrow x^{1+n+1} = 0 \Rightarrow (0 : R_\infty) \subset \mathrm{Nil}(R)$.

Suppose R is noetherian

Let $I = (f_\lambda)$

U_{f_λ} cover $X \iff Z(\{f_\lambda\}) = \emptyset \iff \mathrm{rad}((f_\lambda)) = R_{>0}$.

If $R_{>0}$ is generated by $g_1, \ldots, g_n$ say, then $g_i^N \in I$

any element of $(f_\lambda) \supset R_{>n}$, some n.

T.B.C.

<u>Example</u> $\mathbb{P}^2 = \mathrm{Proj}\, k[x_0, x_1, x_2]$

consider plane curves

a curve can be given by a homogeneous equation f.

Consider 2 curves $Z(f)$, $Z(g)$.

<u>Prop</u> $Z(f) \cap Z(g) \neq \emptyset$. Suppose not.

then $\mathrm{rad}(f, g) = (x_0, x_1, x_2)$. But

$ht(x_0, x_1, x_2) = 3$, so it cannot happen.