Terminology

$R \supset S$ mult. set

$I \subset R$ $\quad I \cap S = \emptyset$

$$S^{-1}R/S^{-1}I \overset{?}{\simeq} \bar{S}^{-1}R/I$$

$\uparrow$ meaning the image of S under $R \to R/I$

Let's use functor of points

other ring

$$\mathrm{Hom}(S^{-1}R/S^{-1}I, Q) = \{\varphi : S^{-1}R \to Q \mid \varphi(I) = 0\}$$

$$= \{\varphi : R \to Q \mid \varphi(I) = 0,\ \varphi(S) \text{ are invertible}\}$$

$$\mathrm{Hom}(S^{-1}R/I, Q) = \{\varphi : R/I \to Q \mid \varphi(S) \text{ are invertible}\}$$

$$= \{\varphi : R \to Q \mid \varphi(S) \text{ are invertible},\ \varphi(I) = 0\}$$

Yoneda's lemma

$$\begin{tikzcd} R \arrow[r] \arrow[dr, bend right, "\substack{I \text{ goes to } 0 \\ S \text{ goes to invertible}}"'] & S^{-1}R/S^{-1}I \arrow[d, dotted, "\exists!"] \\ & Q \end{tikzcd}$$

$\rightsquigarrow$ Uniqueness of the universal property.

$$R_p/m \simeq F(R/p). \qquad m = p_p$$

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by the proof $\mathbb{C}^2 \setminus \{0\}$ is not affine there was a gap.

$$\mathcal{O}(\mathbb{C}^2 \setminus \{0\}) = \mathbb{C}[x,y]$$

if $\mathbb{C}^2 \setminus \{0\}$ is affine ($\simeq$ to an affine scheme as a ringed space)

$\mathbb{C}^2 \setminus \{0\} \simeq \operatorname{Spec} \mathbb{C}[x,y]$ as ringed spaces

we have $(0,0) \in \operatorname{Spec} \mathbb{C}[x,y]$

don't have the notion of "have a point" on the left (yet).

Terminology

$(X, \mathcal{O}_X)$ ringed space ($\Leftrightarrow$ $\mathcal{O}_X$ is a sheaf of rings)

$\mathcal{O}$ is a presheaf:

0) $\mathcal{O}(U)$ is a ring

1) $U \subset V$ we have $p : \mathcal{O}(V) \to \mathcal{O}(U)$ homomorphism of rings

U, V open

sheaf condition:

2) Conditions about covers (injectivity, description of the image $\mathcal{O}(U) \to \prod_{\lambda \in \Lambda} \mathcal{O}(V_\lambda)$ for open covers $\bigcup V_\lambda = U$)

Morphisms of ringed spaces

(Morphisms of schemes are just morphisms of the corresponding ring spaces)

Def a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is

1) continous map $f : X \to Y$

2) $\forall U \subset Y$ open a homomorphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$.

Such that $\forall U \subset V$ open in Y

$$\begin{tikzcd} \mathcal{O}_Y(V) \arrow[r] \arrow[d, "p"] & \mathcal{O}_X(f^{-1}(V)) \arrow[d, "p"] \\ \mathcal{O}_Y(U) \arrow[r] & \mathcal{O}_X(f^{-1}(U)) \end{tikzcd}$$

is commutative.

f is isomorphism if equivalently

1) $X \to Y$ is a homeomorphism, $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is isomorphism

2) $\exists$ inverse morphism f^{-1} s.t. $ff^{-1} = f^{-1}f = Id$

Remark $f^*(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$

$\{\mathcal{O}_X(f^{-1}(U))\}$ is a sheaf of rings on Y denoted by $f_*(\mathcal{O}_X)$, and the data of $\{f^*(U)\}$ is a morphism of sheafs on Y: $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$.

Prop If X is a ringed space and Y is an affine scheme

Morphisms $(X, Y) \longrightarrow$ Ring homomorphisms $\mathcal{O}(Y) \to \mathcal{O}(X)$

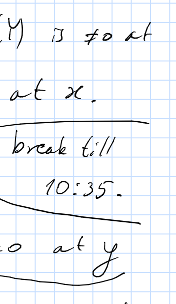
$f : X \to Y$ induces $f^*(Y) : \mathcal{O}(Y) \to \mathcal{O}(X)$

is a bijection. (In particular, if both X, Y are affine then $\mathrm{Mor}(X,Y)$, $\mathrm{Mor}(Y,X)$ are the same as ring homomorphisms)

Proof Given a ring homomorphism $\varphi\ \mathcal{O}(Y) = R \to \mathcal{O}(X)$, let $x \in X$

Given $x \in X$

$f \in \mathcal{O}(X)$ we say that f is $\neq 0$ at $x \in X$ if $\exists\, U \ni x$ such that $f|_U$ is invertible.



$\exists ?\, y \in Y$ s.t. $f \in \mathcal{O}(Y)$ is $\neq 0$ at $y \Rightarrow \varphi(f)$ is $\neq 0$ at x.

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break till 10:35.

$\varphi(f) = 0$ at $x \Rightarrow \underline{f = 0 \text{ at } y}$

$f \in$ prime ideal corresponding to y.

So the question is:

$$\{f \in R \mid \varphi(f) = 0 \text{ at } x\}.$$

Is this a prime ideal? $\forall x \in X$.

Replacing R by $\mathcal{O}(X)$ the question is reduced to a question about X:

Let $x \in X$ is $\{f \in \mathcal{O}(X) \mid f = 0 \text{ at } x\}$ a prime ideal?

By our definition $f = 0$ at $x \Leftrightarrow$ not $f \neq 0$ at $x \Leftrightarrow$ $\forall U \ni x$ open $f|_U$ is not invertible.

So the question is $f|_U$ is not invertible $\forall U \ni x$, $g|_U$ is not invertible $\overset{?}{\Rightarrow} f+g$ is not invertible

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f is not invertible g is any $\Rightarrow$ fg is not invertible

($fgh = 1 \Rightarrow f(gh) = 1$.

So, if X is also a scheme then this is true:

Let $U \ni x$ be an affine scheme. f is not invertible $\forall V \subset U \Rightarrow f \in p_x$.

prime ideal corresponding to x.

Proof: $f \notin p_x \Rightarrow f|_{U_f}$ is invertible $x \in U_f$. $\varnothing$.

Given f, g as before $f + g|_U \in p_x$

if $f+g$ was invertible in some $U' \Rightarrow$ it would be invertible in $\underset{x}{\underset{\cup}{U \cap U'}} \Rightarrow f + g \in p_x$.

Conclusion:

$$\boxed{\begin{array}{l} x \in X \Rightarrow \{f \in \mathcal{O}(X) \mid f \text{ is not invertible in any neighborhood of } x\} \\ \text{is a prime ideal. } p_x \subset \mathcal{O}(X) \end{array}}$$

So given $\varphi : R \to \mathcal{O}(X)$, given $x \in X$ Let $\psi(x) = \varphi^{-1}(p_x)$.

Obtain a map $\psi : X \to \operatorname{Spec}(R) = Y$.

Claim ψ is continuous $\Leftrightarrow$

$\psi^{-1}(\text{open})$ is open $\Leftrightarrow$ $\psi^{-1}(\text{basic open})$ is open $\Leftrightarrow$ $\forall f \in R$ $\psi^{-1}(U_f)$ is open

$$= \{x \mid \underbrace{p_x \not\ni \varphi(f)}_{\varphi(f) \text{ is invertible in some neighborhood of } x}\}$$

clearly open.

finally, given $f \in R$ restriction of f to $\psi^{-1}(U_f)$ is invertible $\Rightarrow \exists!$ map

$$\underset{\mathcal{O}_Y(U_f)}{\underset{\|}{f^{-1}R}} \to \mathcal{O}_X(\psi^{-1}(U_f))$$

by the universal property of localization, Compatible with restrictions:

$$\begin{tikzcd} f^{-1}R \arrow[r] \arrow[d] & \mathcal{O}_X(\ldots) \arrow[d] \\ (fg)^{-1}R \arrow[r] & \mathcal{O}_X(\ldots) \end{tikzcd}$$

Commutative by the univ. property of localization.

Given a ringed space $(X, \mathcal{O}_X)$ for a point $x \in X$ The "local ring at x" is the universal ring $\mathcal{O}_{X,x}$ s.t.

1) $\forall U \ni x$ open $\exists$ homomorphism $\mathcal{O}(U) \to \mathcal{O}_{X,x}$

2) $\forall U \supset V \ni x$ open

$$\begin{tikzcd} \mathcal{O}(U) \arrow[dr] \arrow[d, "p"'] & \\ \mathcal{O}(V) \arrow[r] & \mathcal{O}_{X,x} \end{tikzcd}$$

is commutative.

This means that $\forall$ R ring satisfying 1), 2) instead of $\mathcal{O}_{X,x}$ $\exists!$ homomorphism $\mathcal{O}_{X,x} \to R$ making

$$\begin{tikzcd} \mathcal{O}(U) \arrow[dr] \arrow[d] & \\ \mathcal{O}_{X,x} \arrow[r, "\exists!"'] & R \end{tikzcd}$$

Commutative $\forall U \ni x$ open.

Concretely: take $\bigsqcup_{\substack{U \ni x \\ \text{open}}} \mathcal{O}(U) / \sim$

$f \in \mathcal{O}(U) \sim g \in \mathcal{O}(V)$ is $\exists$ $x \in \underset{\text{open}}{W} \subset U \cap V$ s.t. $f|_W = g|_W$.

addition: $f \in \mathcal{O}(U)$ $g \in \mathcal{O}(V) \to f|_{U \cap V} + g|_{U \cap V}$.

Similarly multiplication.

$\to$ get a ring structure.

Elements of $\mathcal{O}_{X,x}$ are called germs of functions. (Similar to the Taylor series expansion in analysis)

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consider

$$\pi : \mathcal{O}(X) \to \mathcal{O}_{X,x}$$

$\pi(f)$ is invertible $\Leftrightarrow$ f is invertible on some nbh of x.

if $\mathcal{O}_{X,x}$ is a local ring $\Rightarrow$ maximal ideal $= \{f \mid \text{non-invertible}\}$

$\Rightarrow$ we obtain the property that we needed.

Such spaces X are called locally ringed spaces (?).

Problem Fix a field k

Construct a scheme $\mathbb{P}^n$ such that for any field k'

$Mor(\operatorname{Spec} k', \mathbb{P}^n) =$ points of the projective space over k'

$(x_0, \dots, x_n) \in (k')^{n+1} \setminus \{0\} / \sim$.

Exercise $Mor(\operatorname{Spec} k', \mathbb{P}^n)$

$\operatorname{Spec} k' =$ 1 point as a set, so

$$Mor(\operatorname{Spec} k', \mathbb{P}^n) = \{ x \in \mathbb{P}^n, \ \forall U \ni x \text{ open}, \text{ Homomorphism } \mathcal{O}_{\mathbb{P}^n}(U) \to k' \}$$

compatible with restriction.

equivalently $\mathcal{O}_{X,x} \to k'$

equivalently a field embedding $\mathcal{O}_{X,x}/m \to k'$.

Approach 1) gluing:

$(x_0, \dots, x_n) \neq 0 \Rightarrow \exists i$ s.t. $x_i \neq 0 \Rightarrow \exists$ unique representative s.t. $x_i = 1$

$\Downarrow$

open sets U_i covering $\underbrace{\mathbb{P}^n(k)}$ the projective space as a set constructed from k.

$U_i \cong \mathbb{A}^n$.

More generally, if U, V are schemes, suppose $W \subset U$ is open, $W' \subset V$ is open, and $\varphi: W \to W'$ isomorphism is given, then we can construct a scheme $U \sqcup_W V$ such that: $U \subset U \sqcup_W V$ as an open subscheme, and V also.

$$Mor(U \sqcup_W V, X) = \{ f: U \to X, g: V \to X \mid f|_W = g|_{W'} \}$$

identify using φ

$$f|_W = g \circ \varphi.$$

$$Mor(X, U \sqcup_W V) = \left\{ \begin{array}{l} X_U \subset X, \\ X_V \subset X \end{array} \right. \text{ open},$$

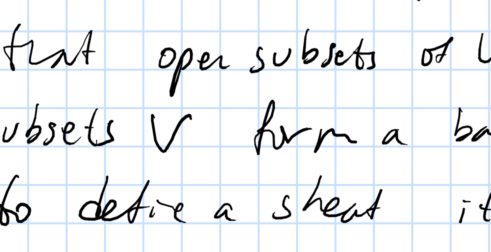
$f: X_U \to U$, $g: X_V \to V$ s.t.

$f(X_U \cap X_V) \subset W$

$g(X_U \cap X_V) \subset W'$

$\varphi \circ f|_{X_U \cap X_V} = g|_{X_U \cap X_V}$. $\}$

Construction: as a set take $U \sqcup V/\sim$ $x \in W$ $x \sim \varphi(x)$.



topology: $S \subset U \sqcup_W V$ is open iff $S \cap U$ & $S \cap V$ are open.

To construct $\mathcal{O}_{U \sqcup_W V}$ we notice that open subsets of U and open subsets V form a basis of topology. to define a sheaf it is enough to define it on sets from the basis.

$S \subset U \Rightarrow \mathcal{O}_{U \sqcup_W V}(S) = \mathcal{O}_U(S)$

$S \subset V$ similarly

$S \subset U \cap V \Rightarrow$ both definitions coincide.

So we obtain a well-defined sheaf of functions.

Application: $\mathbb{P}^1$

Let $U = \mathbb{A}^1$ $\quad V = \mathbb{A}^1$

$W = \mathbb{A}^1 \setminus \{0\}$ $\quad W' = \mathbb{A}^1 \setminus \{0\}$

$\varphi: W \to W'$

$t \to t^{-1}$

Precisely: $U = \operatorname{Spec} k[t]$

$V = \operatorname{Spec} k[t]$

$W = U_t = \operatorname{Spec} t^{-1}k[t]$

$W' = \operatorname{Spec} t^{-1}k[t]$.

$x \in t^{-1}k[t]$ looks like

$c_{-n}t^{-n} + c_{1-n}t^{1-n} + \dots + c_0 + \dots + c_n t^n$.

φ sends t to t^{-1}.

k' points of $\mathbb{P}^1$ $\quad \operatorname{Spec} k' \to \mathbb{P}^1$

correspond to $(k')_1 \sqcup (k')_2 /$ $x \in (k')_1 \neq 0 \Rightarrow (x)_1 \sim (x^{-1})_2$.

From description of maps to $U \sqcup_W V$.

Hope similarly to construct $\mathbb{P}^n$ by gluing $U_i \cong \mathbb{A}^n$.

$\forall$ pair i, j we have U_i, U_j, we know how to glue:

define $W_{ij} \subset U_i$ by $W_{ij} = \{(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots)$ s.t. $x_j \neq 0\}$

So we have $W_{ij} \subset U_i$ $\quad W_{ji} \subset U_j$.

Define $\varphi_{ij}: W_{ij} \to W_{ji}$

$$x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n$$

$$\downarrow \cdot \frac{1}{x_j}$$

$$\left(\frac{x_0}{x_j}, \dots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_{i+1}}{x_j}, \dots, \underset{j}{1}, \dots, \frac{x_n}{x_j} \right)$$

$$\cap$$

$$W_{ji}$$

omit x_i

formally $U_i = \operatorname{Spec} k[x_0, \dots, \hat{x}_i, \dots, x_n]$

$W_{ij} = \operatorname{Spec} x_j^{-1}k[x_0, \dots, \hat{x}_i, \dots, x_n]$

$W_{ji} = \operatorname{Spec} x_i^{-1}k[\dots \hat{x}_j \ ]$

$W_{ij} \to W_{ji}$ corresponds to

$x_i^{-1}k[\_\, \hat{x}_j \,\_] \to x_j^{-1}k[\_\, \hat{x}_i \,\_]$

$x_0 \to \frac{x_0}{x_j}$

$\vdots$

$x_{i-1} \to \frac{x_{i-1}}{x_j}$

$x_i \to \frac{1}{x_j}$ (invertible)

$\vdots$

$x_k \to \frac{x_k}{x_j}$. $(k \neq i, j)$

So we can glue U_i and U_j.

Let's use induction

1) glue $U_1, U_2 \rightsquigarrow U_1 \sqcup_{W_{12}} U_2 = \tilde{U}_2$

$\tilde{U}_2$

$\vee$

we want to glue to U_3:

we need $W \subset \tilde{U}_2$, $W' \subset U_3$

$\varphi: W \to W'$.

$W = W_{13} \cup W_{23}$

Let $W' = W_{31} \cup W_{32} \subset U_3$,

we want to construct a map $W' \to U_1 \sqcup_{W_{12}} U_2$

we need $W_{31} \to U_1$, which we have

$(W_{31} \xrightarrow{\varphi_{31}} W_{13} \subset U_1)$

Similarly $W_{32} \to U_2$.

$W_{31} \cap W_{32}$ should go to W_{12}, W_{21} respectively in U_1, U_2.

so that

$$\boxed{\begin{tikzcd} W_{31} \cap W_{32} \arrow[r, "\varphi_{31}"] \arrow[dr, "\varphi_{32}"'] & W_{13} \cap W_{12} \arrow[d, "\varphi_{12}"] \\ & W_{23} \cap W_{21} \end{tikzcd}}$$

we need this comm. diagram.

T.B.C.