Today: Dimension Theory

Example

Let R be a ring in which every prime ideal is maximal. Assume R Noeterian, reduced.

R maximal ideals $m_1, \dots m_k, \dots$ (may be infinite)

$\bigcap m_i = \{0\}$.

$R \to \underbrace{R/m_1 \times R/m_2 \dots}_{\text{product of fields}}$ injective.

__Prop__ # of maximal ideals is finite.

__Pf__ $\{0\}$ is not prime $\Rightarrow$ $fg = 0$ $f \neq 0$ $g \neq 0$

$\forall i$ $fg = 0 \mod m_i \Rightarrow f = 0 \mod m_i$ or $g = 0 \mod m_i$

$\Rightarrow$ one of them (f or g) $= 0 \mod m_i$ for infinitely many i. Assume f. consider $\text{rad}(f)$, $R/\text{rad}(f)$ again has infinitely many maximal ideals by continue get contradiction. $\lightning$.

$R \to R/m_1 \times \dots \times R/m_k$

__Prop__ This is surjective.

__Pf__ pick i, we construct f which goes to $(0, 0, \dots, 1, 0, \dots 0)$. Let $g_j \in R$ s.t.
pos. i

$g_j = 0 \mod m_j$, $g_j \neq 0 \mod m_i$ (why does it exist? otherwise $m_j \subset m_i$ $\lightning$).

take $\prod_{j \neq i} g_j = g$, we have $g = 0 \mod m_j$ $\forall j \neq i$

$g \neq 0 \mod m_i$ (m_i is prime), so g goes to $(0, \dots, 0, x, 0, \dots 0)$ $x \in R/m_i$, Let $x^{-1} \in R/m_i$ be
pos i represented by $y \in R$ Then

gy goes to $(0, \dots, 1, \dots 0)$, done $\blacksquare$

So $R \cong R/m_1 \times \dots \times R/m_k$ (product of finitely many fields).

__Generalization__

__Def__ (Let R be a ring. Consider $\text{Spec}(R)$.)
A topological space X is irreducible if we cannot have $X = Z_1 \cup Z_2$ for Z_1, Z_2 closed, $Z_1, Z_2 \neq X$.

__Prop__ $\text{Spec}(R)$ is irreducible for R reduced iff R is a domain.

__Pf__ R is a domain $\Rightarrow$ Z_1, Z_2 $Z_1 \cup Z_2 = \text{Spec}(R)$

$\Rightarrow$ $I(Z_1) \cap I(Z_2) = \{0\} \Rightarrow I(Z_1) \cdot I(Z_2) = \{0\}$

$I(Z_1) \neq \{0\} \Rightarrow$ Let $f \in I(Z_1)$ $f \neq 0$

$\forall g \in I(Z_2)$ we have $fg = 0 \Rightarrow g = 0 \Rightarrow I(Z_2) = 0$.

$\Rightarrow Z_1$ or $Z_2 = \text{Spec}(R)$ $\lightning$.

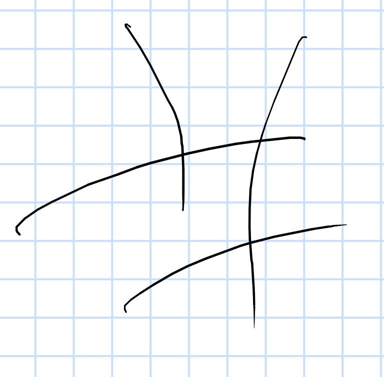
R is not a domain $\Rightarrow$

$\exists f, g \neq 0$ s.t. $fg = 0 \Rightarrow Z(f) \cup Z(g) = 0$

$\text{Spec}(R)$ irreducible $\Rightarrow$ $Z(f) = \text{Spec}(R)$ or $Z(g) = \text{Spec}(R)$

WLOG assume $Z(f) = \text{Spec}(R) \Rightarrow \text{rad}(f) = \{0\} \Rightarrow f = 0$. $\lightning$.

Consider R noetherian. $\text{Spec}(R)$



__Prop__ 1) There is only finitely many maximal irreducible closed subsets of $\text{Spec}(R)$

2) $\text{Spec}(R) =$ union of those.

__Equivalently__ $a \subset R$ ideal is prime $\Leftrightarrow$ $Z(a)$ is irreducible & $a = \text{rad}(a)$.

1) There is only finitely many minimal prime ideals in R $p_1, \dots, p_k$

2) $\{0\} = \bigcap_{i=1}^{k} p_i$

__Pf__ $\{0\}$ is an intersection of finitely many prime ideals: otherwise Construct an increasing sequence $a_1 \subset \dots$ s.t. a_i is not an intersection of finitely many primes., $\text{rad}(a_i) = a_i$

Construction given a_i: a_i is not prime $\Rightarrow \exists f, g \notin a_i$ s.t. $fg \in a_i$. Consider $\text{rad}(a_i + (f))$, $\text{rad}(a_i + (g))$ ($\neq a_i$).

One of them is not an intersection of finitely many primes, because otherwise $\text{rad}(a_i + (f)) \cap \text{rad}(a_i + (g))$

$(a_i + (f))(a_i + (g)) \subset a_i$

$\Downarrow$

$\text{rad}((a_i + (f))(a_i + (g))) \subset \text{rad}(a_i) = a_i$

$\| $

$\text{rad}(a_i + (f)) \cap \text{rad}(a_i + (g))$

clearly $a_i \subset \text{rad}(a_i + (f)) \cap \text{rad}(a_i + (g)) \Rightarrow$ they are equal $\Rightarrow$ contradiction.

So we can choose $a_{i+1} = \text{rad}(a_i + (f))$ or $\text{rad}(a_i + (g))$

Get increasing sequence of ideals $\lightning$. $\blacksquare$

$\{0\} = p_1 \cap p_2 \dots \cap p_k$ some prime ideals p_i.

__Claim__ Take p_i, suppose p_i is not minimal prime

$\Rightarrow$ $p_i \supset q$ (prime q) $q \supset \{0\}$

$q \supset p_1 \cap p_2 \dots p_k \Rightarrow q \supset p_1 \dots p_k$, since q is prime

$q \supset p_j$ some p_j. $j \neq i$ $\Rightarrow$ $p_i \supset q \supset p_j$

$\Rightarrow$ $\{0\} = p_1 \cap \dots p_{i-1} \cap p_{i+1} \cap \dots p_k$.

Continue until we get $\{0\} = p_1 \cap \dots \cap p_k$ and $p_1, \dots, p_k$ are minimal prime ideals.

Let q be a minimal prime of R $\Rightarrow$

$\{0\} = \bigcap p_i \subset q \Rightarrow p_i \subset q$ some $i \Rightarrow p_i = q$.

$\Rightarrow$ So all minimal prime ideals are among $p_1, \dots, p_k$ we are done. $\blacksquare$

__Cor__ In general, R noetherian $\text{rad}(\{0\}) = \text{Nil}(R) \Rightarrow$ $\exists$ finitely many minimal prime ideals $p_1, \dots, p_k$ $\bigcap p_i = \text{Nil}(R)$

also $a \subset R$ ideal $\Rightarrow$ the set of primes containing a has finitely many minimal elements $p_1, \dots p_k$, $\text{rad}(a) = p_1 \cap \dots p_k$.

Ex $R = k[x_1, \dots, x_n]$ (k field)

$f \in R$, $f \neq 0$.

R is a U.F.D. $\Rightarrow$ $f = f_1^{r_1} \dots f_k^{r_k}$ for

$f_1, \dots, f_k$ irreducible. $\Rightarrow$

$(f) = (f_1)^{r_1} \dots (f_k)^{r_k} \Rightarrow$

$\mathrm{rad}(f) = \mathrm{rad}(f_1) \cap \dots \mathrm{rad}(f_k)$,

$f_i = \mathrm{rad}(f_i)$ is prime (again, using U.F.D.)

$\Rightarrow \mathrm{rad}(f) = (f_1) \cap \dots \cap (f_k)$ $\quad (f_i) \not\subset (f_j)$ if $i \neq j$

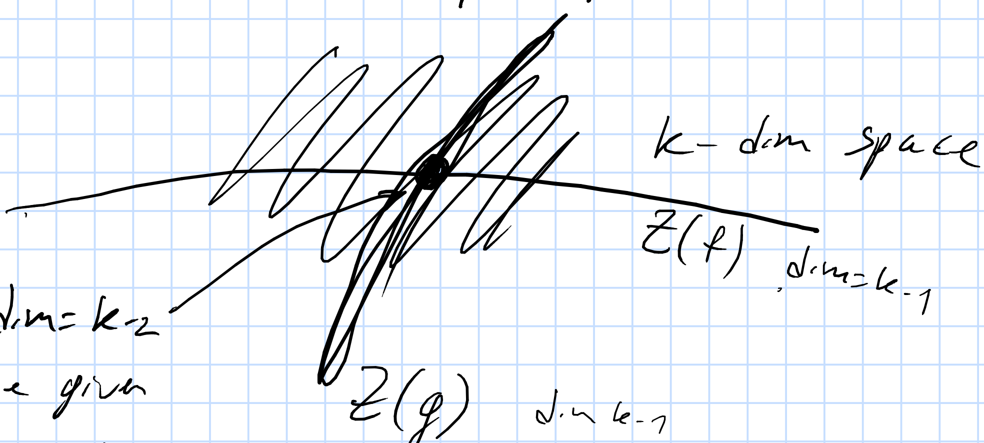
prime ideals

$\Rightarrow$ (f_i) are the minimal prime ideals containing (f).

So all minimal $\neq 0$ primes are principal.

(Exercise: if all minimal $\neq 0$ primes are principal $\Rightarrow$ R is U.F.D.)

$R = k[x_1, \dots, x_k]$



Ideal $Z(f,g)$ cannot be given by a single equation.

Some definitions:

if $p \subset R$ is prime then

Def 1 $\mathrm{height}(p) = \max\{k \mid \exists_{\text{primes}}\ p_0 \subsetneq p_1 \subsetneq \dots \subsetneq p_k = p\}$ (can be ∞)

p minimal $\Leftrightarrow$ $\mathrm{height}(p) = 0$

in $k[x_1, \dots, x_k]$ $\quad \mathrm{height}(p) = 1 \Leftrightarrow p = (f)$ (f irreducible)

Geometrically $\mathrm{height}(p) = \dim \mathrm{Spec}(R) - \dim \mathrm{Spec}(R/p)$.

Def 2 $\dim(R) = \max_{p \text{ prime}} \mathrm{height}(p)$.

$= \max\{k \mid \exists_{\text{primes}}\ p_0 \subsetneq \dots \subsetneq p_k \subsetneq R\}$.

Def 3 X top. space

$\dim X = \max\{k \mid \exists_{\substack{\text{irreducible} \\ \neq \emptyset \text{ closed}}}\ Z_0 \subsetneq Z_1 \subsetneq \dots \subsetneq Z_k \subset X$

Prop $\dim \mathrm{Spec}(R) = \dim R$.

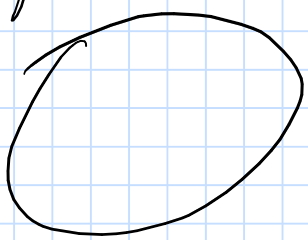
Def 4 $g(p) = \{\min(k) \mid p$ appears as a minimal prime containing $(f_1, \dots, f_k)\ \}$

metric spaces Hausdorff $\longleftrightarrow$ "combinatorial spaces" with finitely many points

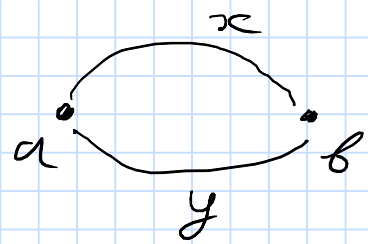
"theorem" any top space is weakly equivalent to a combinatorial space.

Example

S^1



Space $\{a, b, x, y\} = X$



S^1 is weakly equivalent to X

topology: $\{a\}, \{b\}$ closed

$$\overline{\{y\}} = \{a, b, y\}$$

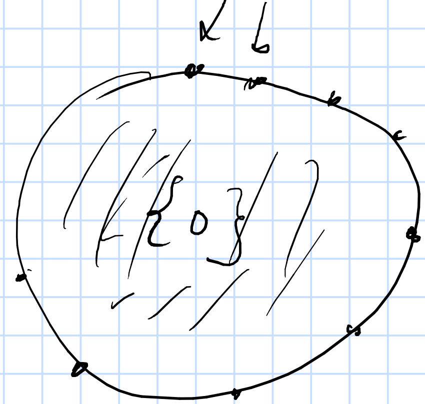
$$\overline{\{x\}} = \{a, b, x\}$$

$$\pi_1(X) = \mathbb{Z}$$

$$C(S^1, X) = \left\{ \begin{array}{l} S^1 = A \sqcup B \sqcup X \sqcup Y \\ \text{s.t. } A, B \text{ closed} \\ \quad X, Y \text{ open} \end{array} \right\}$$

$$\mathbb{A}^1 = \operatorname{Spec} k[x]$$

maximal ideals



Ringed spaces (cont.)

Write def again $\mathcal{O} = \mathcal{O}_X$

X top space $\forall U \subset X$ open $\mathcal{O}(U)$ a ring

$\forall U \subset V \subset X$ open $\rho_{U,V} : \mathcal{O}(V) \to \mathcal{O}(U)$ a ring homo.

Axioms: $U \subset V \subset W \subset X$

1) $\rho_{U,W} = \rho_{U,V} \circ \rho_{V,W}$ $\quad \prod \rho_{U_\lambda, U}$

2) $U = \bigcup_{\lambda \in \Lambda} U_\lambda \Rightarrow \mathcal{O}(U) \to \prod_{\lambda \in \Lambda} \mathcal{O}(U_\lambda)$ is injective

3) if $\{f_\lambda \in \mathcal{O}(U_\lambda)\}$ satisfy $\forall \lambda, \mu$ $\rho_{U_\lambda \cap U_\mu, U_\lambda}(f_\lambda) = \rho_{U_\lambda \cap U_\mu, U_\mu}(f_\mu)$ then $\{f_\lambda\}$ is in the image of 2).

Example $\operatorname{Spec}(R)$ is a ringed space.

How to define $\mathcal{O}(U)$?

Sets U_f $(f \in R)$ form a basis of topology.

Any open U is a union of U_{f_λ} $f_\lambda \in R$

by 2) 3) we will recover $\mathcal{O}(U)$. $(U_f \cap U_g = U_{fg})$

So let's define $\mathcal{O}(U_f)$ $f \in R$

$U_f \simeq \operatorname{Spec}(R[x]/(fx-1))$ so set

$\mathcal{O}(U_f) = R[x]/(fx-1)$.

Restriction. Suppose $U_f \subset U_g$ we need $\mathcal{O}(U_g) \to \mathcal{O}(U_f)$

$\frac{1}{g} \to ?$ $\quad f^{-n} h$

$$\begin{aligned} Z(f) &\supset Z(g) \\ &\Updownarrow \\ IZ(f) &\subset IZ(g) \\ &\Updownarrow \\ \operatorname{rad}(f) &\subset \operatorname{rad}(g) \\ &\Updownarrow \\ f &\in \operatorname{rad}(g) \\ f^n &= 0 \mod g \\ f^n &= g \cdot h \quad h \in R \\ \frac{1}{g} &= f^{-n} h \end{aligned}$$

$R[x]/(gx-1) \to R[x]/(fx-1)$

$x \to x^n h$

Since $gx - 1 \to gx^n h - 1 = f^n x^n - 1 = 0$ the homomorphism is well-defined.

In fact $\mathcal{O}(U_g) \overset{\varphi}{\dashrightarrow} \mathcal{O}(U_f)$, $R \to \mathcal{O}(U_g)$, $R \to \mathcal{O}(U_f)$ $\quad \varphi$ is unique making diagram commutative

because choice of φ corresponds to a choice of $\alpha \in \mathcal{O}(U_f)$ s.t. $\alpha g = 1$, inverse is unique.

For $U = \bigcup_{\lambda \in \Lambda} U_{f_\lambda}$

$\mathcal{O}(U) = \left(\alpha_\lambda \in \mathcal{O}(U_{f_\lambda})\right)_{\lambda \in \Lambda}$ s.t. $(\forall \lambda, \mu \ \ldots)$

this defines a ring.

Problem what about idependence of $\{U_{f_\lambda}\}$?

Solution Let $\Lambda = \{f \mid U_f \subset U\}$

$\lambda \in \Lambda$ $f_\lambda = \lambda$.

Solve the uniqueness problem.

for $U = U_f$ we have 2 definitions.

$\mathcal{O}(U_f) \xrightarrow[\text{restrict}]{} (f_\lambda \in \mathcal{O}(U_\lambda))$ $U_\lambda \subset U_f$

$\xleftarrow[\text{take } \lambda = f]{}$

The main problem is to prove 2) & 3) for

$\boxed{U_f = \bigcup_{\lambda \in \Lambda} U_{g_\lambda}}$

$Z(f) = \bigcap Z(g_\lambda)$

$\operatorname{rad}(f) = \sum \operatorname{rad}(g_\lambda)$

$g_\lambda^{n_\lambda} = 0 \mod f$

$f^n \in \sum \operatorname{rad}(g_\lambda)$

$f^n = a_1 + \ldots a_k$ $\quad \lambda_1, \ldots, \lambda_k \in \Lambda$

$a_i \in \operatorname{rad} g_{\lambda_i}$

We have $a_i^m \in g_{\lambda_i}$ some m

$\Leftarrow f^{mnk} \in (g_{\lambda_1}, \ldots, g_{\lambda_k})$

$\mathcal{O}(U_f) \to \prod \mathcal{O}(U_{g_\lambda})$

$R[f^{-1}]$

t.b.c.