Commutative algebra WS18
Exercise set 10.

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Problem 1. [AM, Ch. 3, Ex. 11] Let R be a ring. Prove that the following are equivalent:

(1) $R/\operatorname{Nil}(R)$ is absolutely flat.
(2) Every prime ideal of R is maximal.
(3) $\operatorname{Spec}(R)$ is a T_1-space (every subset consisting of a single point is closed.
(4) $\operatorname{Spec}(R)$ is Hausdorff.

Problem 2. [AM, Ch. 3, Ex. 16] Let R' be a flat R-ring via a homomorphism $\varphi : R \to R'$. If I is an ideal of R, then $R'\varphi(I)$ denotes the ideal of R' generated by $\varphi(I)$. Show that the following conditions are equivalent:

(1) For all ideals $I \subset R$, we have $\varphi^{-1}(R'\varphi(I)) = I$.
(2) $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ is surjective.
(3) For every maximal ideal $\mathfrak{m}$ of R, we have $R'\varphi(\mathfrak{m}) \neq R'$.
(4) For every non-zero R-module M, we have $R' \otimes_R M \neq 0$.
(5) For every R-module M, the natural mapping $M \to R' \otimes_R M$ is injective.

In such a situation, R' is said to be faithfully flat over R.

Problem 3. [AM, Ch. 3, Ex. 17] Let $R \xrightarrow{f} R' \xrightarrow{g} R''$ be ring homomorphisms. Show that if $g \circ f$ is flat and g is faithfully flat, then f is flat.

Problem 4. [AM, Ch. 3, Ex. 25] Let $f : R \to R'$ and $g : R \to R''$ be ring homomorphisms. Show that there is a ring structure on $R' \otimes_R R''$ such that for any $x_1, x_2 \in R'$ and $y_1, y_2 \in R''$ we have

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1x_2 \otimes y_1y_2.$$

Let $h : R \to R' \otimes_R R''$ be the homomorphism defined by $x \to x \otimes 1$ (The AM book has a mistake here). Show that

$$h^* \operatorname{Spec}(R' \otimes_R R'') = f^* \operatorname{Spec}(R') \cap g^* \operatorname{Spec}(R'').$$

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